EVOLUTIONARY CONSUMERS IMPLY MONOPOLIES EXIT*

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Abstract. We address the question of how a monopolist should price when facing evolutionary consumers who gradually move in the direction of following their optimal strategy but may make temporary suboptimal choices. We show that under a broad generalization of the most commonly used model of evolution, the monopolist will set a path of prices such that all consumers eventually stop purchasing the monopolist's product.

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1. Introduction

While a standard rationalist paradigm assumes that individuals optimize instantaneously, the assumption of full rationality on the part of economic agents has been widely questioned in the literature (see Camerer *et al.* 2004 for an extensive collection of essays related to this). For most people, optimizing is costly, and even with good intentions, mistakes are made. Many people are not on the optimal cell phone plan given their usage, use credit cards with dominated terms (Ausubel 1991), and don't save enough for retirement (Bernheim *et al.* 2001). Product adoptions often follow an "S-shaped" pattern, reflecting a period in which the adoption rate increases as awareness increases, followed by a diminishing rate as the set of people available to adopt dwindle.

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The fact that the assumption of full rationality may not be satisfied has prompted an examination of alternative models. Evolutionary models have been a particularly popular choice, as they provide a compelling framework for understanding how people actually behave. These models typically have the property that individual decisions are eventually optimal, and in a static environment, improve over time. Evolutionary models have been extensively studied as a model of game play since Maynard Smith and Price (1973) introduced the concept of evolutionarily stable strategies, and the literature is generally supportive of their use in modeling game play. For instance, evolutionary models have been applied to a wide range of problems such as crime (Cressman *et al.* 1998), firm market shares (Mazzucato 1998), industrial dynamics (Klarl 2008), livestock management (Gramig and Horan 2011), portfolio selection (Bomze 2000), the prisoner's dilemma (Epstein 1999), public goods (Brandt *et al.* 2006), technological innovation (Windrum 1999), tourism (Accinelli *et al.* 2009), and value chains (Cantner *et al.* 2016).

As noted in Weibull (1998a), the most widely used model of evolution in the literature is the replicator dynamics, which was first developed by Taylor and Jonker (1978).[2] Under the replicator dynamics, the rate at which the fraction of players who employ a particular strategy changes is directly proportional to the difference between the average payoff obtained by players who employ that strategy and the average payoff of all the players. Such evolutionary models based on the replicator dynamics have been used successfully in a myriad of applications in the literature. Not only is there an extensive economics literature based on

[2]Cressman and Tao (2014) also note that "the replicator equation is the first and most important game dynamics studied in connection with evolutionary game theory."

such models,[3] but these models have also been used extensively in biology[4] as well as dozens of other scientific disciplines.[5] Frequently the behavior of firms is modeled this way (*e.g.* Cantner *et al.* 2016; Klarl 2008; Mazzucato 1998), but the replicator dynamics could also be applied to consumers.

We will ask a practical question: how should a monopolist price when facing such evolutionary consumers? A monopolist faces a tradeoff: an increased price today will result in greater immediate revenues, but at a cost of consumers leaving the service at a faster rate. The evolutionary framework is attractive because it both accords well with how real consumers appear to behave, and because that behavior – temporary suboptimal choices – seems to accord well with how some managers think about their pricing decisions.

We start with the simplest possible model in which a monopolist is selling a service to a continuum of consumers with unit demand at a common value, whose evolution is governed by the replicator dynamics.[6] Consumers can shut off their service, or sign up, at any

[3]Accinelli *et al.* 2009; Bendor and Swistak 1998; Binmore *et al.* 1995; Bomze 2000; Börgers and Sarin 1997; Boylan 1994; Branch and McGough 2008; Brandt *et al.* 2006; Cabrales 2000; Cabrales and Sobel 1992; Cheng *et al.* 2004; Cheung and Friedman 1998; Cressman *et al.* 1998; Dekel and Scotchmer 2000; Eichberger *et al.* 1993; Ely and Sandholm 2005; Epstein 1999; Friedman 1998; Fudenberg and Harris 1992; Gaunersdorfer and Hofbauer 1995; Gramig and Horan 2011; Kim 1996; Maliath 1998; Mazzucato 1998; Oechssler and Riedl 2001, 2002; Pawlowitsch 2008; Samuelson and Zhang 1992; Sandholm 2008, 2009; Sandholm *et al.* 2008; Sigmund *et al.* 2011; Viossat 2007; Weibull 1995, 1998a,b; Windrum 1999; Young and Foster 1991.

[4]Alboszta and Miękisz 2004; Benaïm *et al.* 2008; Bomze 1983, 1995; Boyd and Richerson 2002; Foster and Young 1990; Fudenberg *et al.* 2006; Hauert 2010; Hauert *et al.* 2002, 2004; Hilbe 2011; Luthi *et al.* 2009; Nowak and Sigmund 2004; Sasaki and Unemi 2011; Stadler and Stadler 2003; Van Veelen 2011.

[5]In a working paper version of this manuscript, we note that the replicator dynamics has been used in computer vision, ecology, genetics, machine learning, mathematics, neuroscience, optimization, philosophy, physics, political science, routing, and spectrum sensing. See Hummel and McAfee (2015) for references.

[6]Surprisingly, we are not finding this model in the literature. The closest models we have found are those in Radner (2003) and Radner and Richardson (2003). These papers also consider evolutionary models in which a monopolist sells to consumers with common values, and find similar substantive conclusions to those in our model with homogeneous consumers. However, the evolution of consumers in these models is not governed by the replicator dynamics. Also see Radner *et al.* (2014).

time. The model is a reasonable model of a cable service, cell phone provider, or insurance provider, although it ignores the issue of contracts and switching costs.[7] The monopolist is forward-looking and starts with an arbitrary initial market share representing the fraction of consumers buying the good.

We find that there is a unique limiting market share for the monopolist, and the monopolist achieves that market share almost immediately. That is, if the monopolist's initial market share is too high, it will charge a very high price for a very short period of time to drive just the right number of consumers away before then charging a price equal to the consumers' common value. Similarly, if the monopolist's initial market share is too low, the monopolist runs a short-lived sale to bring in the necessary contingent of consumers. This limiting market share decreases in the monopolist's discount rate and increases in the speed at which consumers evolve, which are not controversial comparative statics.

We then consider how a more sensible specification of demand affects the model. We permit a variety of consumer types, each with a distinct value for the product. The monopolist cannot price discriminate but can vary price over time. Market shares will evolve differently for different types of consumers because different consumers have different values for the product. Throughout we assume consumers evolve according to some general model that subsumes the replicator dynamics as a special case.

We prove that the monopolist eventually sells only to the highest type, and the price converges to the value of the highest type. In particular, if there is a continuum of types, the fraction of consumers buying the product goes to zero, and the monopolist eventually exits. This is true under very general conditions about the decisions made by consumers.

[7]Implicit in the replicator dynamics formulation is that consumers only change their mind about whether to purchase the product upon meeting some other individual who is making the opposite decision. This seems reasonable with insurance, for instance, where gathering information from others may instigate one to make a different decision about whether to buy insurance. It also seems plausible for cable TV and cell phones, where consumers who are addicted to using these products may only realize they would be better off without them upon encountering someone who is happier after ceasing to use these products.

We argue in the conclusion that this property—that a monopolist eventually exits—makes the replicator dynamics unsuitable for describing the long-run evolution of consumers. Moreover, the problem with this model is that people are more responsive to prices than the model posits. We defer this argument until after the analysis. First we present the analysis of the one type case.

1.1. **The One Type Case.** There is a continuum of consumers whose value for a product, net of opportunity costs, is v. We let $x(t)$ denote the monopolist's market share at time t, where $x(0)$ is given. The firm charges a price $p(t)$ at time t, and the only restriction imposed on the price path is that the Riemann integral $P(t) = \int_0^t p(s)\ ds$ is well-defined. Consumers may adopt or stop using the product at will, and we assume that the fraction of consumers who adopt the product evolves according to the replicator dynamics. The replicator dynamics requires that the rate at which this fraction changes is proportional to the average utility of adopters, minus the average utility, times the fraction of adopters x. Since the instantaneous utility of adoption is $v - p$, while the average utility of consumers is $(v - p)x$, under the replicator dynamics we have

$$\dot{x} = \lambda((v-p) - (v-p)x))x = \lambda(v-p)x(1-x).\qquad (1)$$

[8]

Now let c denote the monopolist's cost per product, which we assume to be constant, and let r denote the monopolist's discount rate. We can then write the seller's profits as

$$\begin{aligned}
\Pi &= \int_0^\infty e^{-rt}(p-c)x\ dt = \int_0^\infty e^{-rt}(v-c)x\ dt - \int_0^\infty e^{-rt}(v-p)x\ dt \\
&= \int_0^\infty e^{-rt}(v-c)x\ dt - \frac{1}{\lambda}\int_0^\infty e^{-rt}\frac{\dot{x}}{1-x}\ dt \\
&= \int_0^\infty e^{-rt}(v-c)x\ dt + \frac{r}{\lambda}\int_0^\infty e^{-rt}\log(1-x)\ dt + \frac{1}{\lambda}e^{-rt}\log(1-x)|_{t=0}^\infty \\
&= \int_0^\infty e^{-rt}\left((v-c)x + \frac{r}{\lambda}\log(1-x)\right)\ dt - \frac{1}{\lambda}\log(1-x(0)).
\end{aligned}$$

[9]

[8] In games with two strategies, it is already known that under the replicator dynamics, the fraction of players employing a strategy changes at a rate proportional to the product of the utility difference between the two strategies and the fraction of players employing each strategy. See *e.g.* Weibull (1995).

The function $(v-c)x+\frac{r}{\lambda}\log(1-x)$ is strictly concave in x and can be maximized pointwise over $x \in [0,1]$ at the value $x^* = \max\{0, 1 - \frac{r}{\lambda(v-c)}\}$. This in turn implies that the firm maximizes its profits by choosing prices in such a way that $x(t)$ is as close to x^* as possible. Thus if $x(0) < x^*$, then it is optimal for the firm to set a price of $p(t) = 0$ until $x(t)$ increases to x^* and then set a price of $p(t) = v$ going forward. Similarly, if $x(0) > x^*$, then it is optimal for the firm to set the maximum possible price until $x(t)$ decreases to x^* and then set a price of $p(t) = v$ going forward. The firm achieves a higher market share when the firm is more patient (r is low), the gains from trade $(v-c)$ are larger, or consumers adjust more rapidly (λ is larger).

The single type model appears to offer an attractive pricing model for a monopolist. The model is also robust to a variety of restrictions. Note that equation (1) is solvable in closed form, as $\frac{d}{dt}\log(\frac{x}{1-x}) = \frac{\dot{x}}{x(1-x)} = \lambda(v-p)$, so that $\log(\frac{x(t)}{1-x(t)}) - \log(\frac{x(0)}{1-x(0)}) = \lambda(vt - P(t))$, or

$$\frac{x(t)}{1-x(t)} = \frac{x(0)}{1-x(0)}e^{\lambda(vt-P(t))}. \tag{2}$$

Unfortunately equation (2) demonstrates that there are going to be problems if there are multiple types. If we make the dependence on v explicit in equation (2), so that

$$\frac{x(v,t)}{1-x(v,t)} = \frac{x(v,0)}{1-x(v,0)}e^{\lambda(vt-P(t))},$$

then

$$\frac{\frac{x(v,t)}{1-x(v,t)}}{\frac{x(u,t)}{1-x(u,t)}} = \frac{\frac{x(v,0)}{1-x(v,0)}}{\frac{x(u,0)}{1-x(u,0)}}e^{\lambda(v-u)t}.$$

This ratio doesn't depend on the price path and it diverges or converges to zero as t gets large. Thus, every price path has the property that, if type v_0 has a market share headed to 1, any type $v > v_0$ not only has a market share closer to 1, but arbitrarily closer to 1 after

[9] Note that $\lim_{t\to\infty} e^{-rt}\log(1-x(t)) = 0$ since $\dot{x} \leq \lambda v(1-x)$, which implies $|\log(1-x(t))| \leq \lambda vt + C$ for some constant C and $\lim_{t\to\infty} e^{-rt}\log(1-x(t)) = 0$. The firm's profits are increasing in $x(0)$, but they do not diverge as $x(0) \to 1$ if there is a maximum price p_{max} because then the firm's profits can never exceed $\frac{p_{max}-c}{r}$.

a sufficiently long time. This will, in turn, imply that those consumers are secure to price increases, which makes price increases inevitable in the model, thereby eventually driving consumers out of the market. Thus monopolists eventually exit when consumers' values are drawn uniformly from the interval $[v - \epsilon, v + \epsilon]$ for some small $\epsilon > 0$ even though the monopolist sells forever when all consumers' values are v.

In the next section, we consider a generalization of this model and prove the main theorem.

2. Model

There is a monopolist firm and a continuum of possible consumer types $\Theta = [\underline{\theta}, \overline{\theta}]$ that is a closed and bounded subset of the non-negative reals. Throughout we let $\theta \in \Theta$ denote an arbitrary consumer type, which will determine a consumer's willingness to pay for the product. Consumer types θ are distributed according to the cumulative distribution function $\Gamma(\theta, t)$ with corresponding probability density function $\gamma(\theta, t)$ that varies continuously with the time t and satisfies $\gamma(\theta, t) \in [\gamma_L, \gamma_H]$ for some $\gamma_L > 0$ and $\gamma_H < \infty$ for all $\theta \in \Theta$ and t.

A consumer of type θ is willing to pay some amount $v(\theta, t)$ for a product being sold by the firm at time t, where $v(\theta, t)$ is a strictly increasing function in θ for all t, and $\overline{v} \equiv \sup_t v(\overline{\theta}, t) < \infty$. Throughout we also assume that $\tilde{v}(q) \equiv \inf_{\theta,t} v(\theta + q, t) - v(\theta, t) > 0$ for all $q > 0$. Thus the maximum value of any of the consumers remains finite and bounded for all t and the difference in values between any two consumer types also remains bounded away from zero for all t.

There are an infinite number of consumers of any given type θ, and the fraction of consumers of type θ who consume the product at time t is some quantity $x(\theta, t)$. The fraction $x(\theta, t)$ evolves according to a process that may depend on a number of different variables. In general we allow the rate at which this fraction changes to depend on the difference between the consumer's willingness to pay for the product and the product's price, the fraction of consumers of this type who are already consuming the product, and some weighted average of the fraction of the other types who are consuming the product.

Formally, we let $p(t)$ denote the price that the firm sets at time t, and let $\bar{x}(t) = \int_{\underline{\theta}}^{\overline{\theta}} w(\theta,t)x(\theta,t)\ d\theta$ denote a weighted average of the values of $x(\theta,t)$ according to some weighting term $w(\theta,t)$ that satisfies $\int_{\underline{\theta}}^{\overline{\theta}} w(\theta,t)\ d\theta = 1$ for all t and varies continuously with t. An example weighting function would be $w(\theta,t) = \gamma(\theta,t)$, in which case $\bar{x}(t) = E_{\theta}[x(\theta,t)]$ for all t.

At time $t = 0$, the value of $x(\theta,0)$ is an exogenously given function of θ in the interval $(0,1)$ that varies continuously with θ. For times $t \geq 0$, if $p(t)$ denotes the price that the firm sets at time t, then $\dot{x}(\theta,t) = f(v(\theta,t) - p(t))g(x(\theta,t))h(\bar{x}(t))$, where $f(\cdot)$ is an arbitrary locally Lipschitz continuous and strictly increasing function satisfying $f(0) = 0$, $g(\cdot)$ is an arbitrary Lipschitz continuous function on $[0,1]$ that satisfies $g(x) > 0$ for all $x \in (0,1)$ and $\lim_{x\to 0} g(x) = \lim_{x\to 1} g(x) = 0$, and $h(x)$ is an arbitrary locally Lipschitz continuous and strictly positive function throughout the interval $[0,1]$. Thus the rate at which the fraction of consumers of type θ who are consuming the product changes can depend on the difference between the consumer's willingness to pay for the product and the product's price $(v(\theta,t) - p(t))$, the fraction of consumers of this type who are already consuming the product $(x(\theta,t))$, and a weighted average of the fraction of consumers of all types who are consuming the product $(\bar{x}(t))$. The model here is far more general than standard models of the replicator dynamics, which effectively assume that $f(v) = \lambda v$ for some $\lambda > 0$, $g(x) = x(1-x)$, and $h(x) = 1$, though it is neither more general nor less general than the regular selection dynamics in Hofbauer and Weibull (1996) and Riztberger and Weibull (1995).

In this model the assumption that $f(\cdot)$ is a strictly increasing function satisfying $f(0) = 0$ implies that the fraction of consumers of a given type who consume a product increases (decreases) if the product is being sold for less (more) than that type's willingness to pay, and the rate at which this fraction increases (decreases) is increasing in the magnitude of the difference between a type's willingness to pay and the product's price. The assumption that $\lim_{x\to 0} g(x) = \lim_{x\to 1} g(x) = 0$ implies that if nearly all consumers of a given type are already consuming (not consuming) the product, then the rate at which this fraction of consumers will continue to increase (or decrease) becomes arbitrarily small.

The firm's profit from a given pricing strategy $p(t)$ is $\int_0^\infty \delta^t (p(t) - c) E_\theta[x(\theta, t)]\, dt$, where $\delta \in (0, 1)$ denotes the firm's discount factor and $c \geq 0$ denotes the cost of the product to the firm. We seek to analyze the properties of a firm's optimal pricing strategy in this model of evolutionary consumers. Throughout we assume that the firm must always set a non-negative price and there is some maximum possible price $P > \overline{v}$ such that a firm may never set a price greater than P.

3. Results

We ultimately seek to show that under the optimal pricing strategy, it is necessarily the case that the fraction of consumers who purchase the product tends to zero in the limit as $t \to \infty$. To prove this, we derive several preliminary results on how the fraction of consumers who purchase the product must evolve over time. First we present the following lemma:

Lemma 1. *There exists some continuous and strictly increasing function ϕ satisfying $\lim_{x \to -\infty} \phi(x) = 0$ and $\lim_{x \to \infty} \phi(x) = 1$ such that the fraction of consumers of type θ who purchase the product at time T, $x(\theta, T)$, satisfies $x(\theta, T) = \phi(\phi^{-1}(x(\theta, 0)) + \int_0^T f(v(\theta, t) - p(t)) h(\bar{x}(t))\, dt)$.*

All proofs are in the appendix. This result shows that the fraction of consumers of a given type who purchase a product at time T, $x(\theta, T)$, can be written as a function of the fraction of consumers who purchased this product at time $t = 0$, $x(\theta, 0)$, and the integral $\int_0^T f(v(\theta, t) - p(t)) h(\bar{x}(t))\ dt$. The result is proven by rewriting the equation $\frac{dx(\theta,t)}{dt} = f(v(\theta, t) - p(t)) g(x(\theta, t)) h(\bar{x}(t))$ as $\frac{dx(\theta,t)}{g(x(\theta,t))} = f(v(\theta, t) - p(t)) h(\bar{x}(t))\, dt$, integrating both sides, and rearranging terms to obtain an expression of the form in Lemma 1.

A consequence of this result is that if a significant fraction of consumers are still purchasing the product even after a long time has passed, then it will necessarily be the case that the fraction of some consumer types who purchase the product will be exceedingly close to 1. In particular, we have the following result:

Lemma 2. *For any value of $\epsilon > 0$, there exists some sufficiently large T such that the following results both hold:*

(a). If $x(\theta^*, t^*) = y$ *for some* $t^* \geq T$ *and* $y \in (\epsilon, 1-\epsilon)$*, then* $x(\theta, t) < \epsilon$ *if* $\theta \leq \theta^* - \epsilon$ *and* $x(\theta, t) > 1 - \epsilon$ *if* $\theta \geq \theta^* + \epsilon$ *for all* $t \in [t^*, t^* + \frac{\log(\epsilon)}{\log(\delta)}]$ *regardless of the price path chosen by the firm.*

(b). If $x(\underline{\theta}, t^*) > \epsilon$ *for some* $t^* \geq T$*, then* $x(\theta, t) > 1-\epsilon$ *if* $\theta \geq \underline{\theta} + \epsilon$ *for all* $t \in [t^*, t^* + \frac{\log(\epsilon)}{\log(\delta)}]$ *regardless of the price path chosen by the firm.*

Lemma 2 follows directly from Lemma 1. If prices up to some long time T have been such that a significant fraction of consumers of type θ are consuming the product, then consumers who value the product slightly more than this will have evolved to the point where nearly all of them are consuming the product. But once nearly all consumers of a given type are purchasing the product, then the rate at which the fraction of consumers of this type who purchase the product changes becomes arbitrarily small since $\lim_{x \to 1} g(x) = 0$. Thus even if the firm charges the maximum possible price, nearly all of these consumers will still be purchasing the product after a very large amount of additional time has passed.

Similar reasoning implies that if prices up to some long time T are such that a significant fraction of consumers of type θ are not consuming the product, then even if the firm charges nothing, nearly all consumers who value the product for less than θ will not purchase the product even after a very large amount of additional time has passed. Together, these insights imply the result in Lemma 2.

An immediate consequence of Lemma 2 is that if a significant fraction of consumers are still purchasing the product even after a very long time has passed, then the firm will want to charge close to the maximum possible price for a considerable amount of time thereafter. In particular, we have the following result:

Lemma 3. *For any fixed* $y \in (0,1)$ *and fixed* $\theta^* < \overline{\theta}$*, consider some arbitrarily small value of* $\epsilon > 0$*, and let* $T_d(\epsilon)$ *denote a value of* T *corresponding to the value of* ϵ *that satisfies the conditions given in Lemma 2. Then if* $x(\theta, t^*) \geq y$ *for some* $\theta \leq \theta^*$ *and some* $t^* \geq T_d(\epsilon)$*, then the firm maximizes its profits from the game beginning at time* t^* *by choosing prices within* $O(\epsilon)$ *of the maximum possible price for all* $t \in [t^*, t^* + \frac{\log(\epsilon)}{\log(\delta)}]$*.*

Lemma 3 follows from Lemma 2. Since Lemma 2 guarantees that the fraction of consumers who will purchase a product going forward will hardly be affected by the prices that a firm charges once a considerable amount of time has passed, a firm might as well charge prices that are exceedingly close to the maximum possible price that the firm can ever charge. This implies the result in Lemma 3.

But Lemma 3 ultimately implies that a price path in which a positive fraction of consumers continue to purchase the product as $t \to \infty$ is inconsistent with profit-maximizing behavior. We present the main result of this section below:

Theorem 1. *Under the optimal price path, the fraction of consumers who purchase the product tends to zero in the limit as* $t \to \infty$.

The reason for this result is that if the fraction of consumers who purchase the product does not tend towards zero in the limit as $t \to \infty$, then we know from Lemma 3 that the firm will have an incentive to charge the maximum possible price. But if the firm charges such a large price, then ultimately the fraction of consumers who purchase the product will have to tend towards zero. This gives the result in Theorem 1.[10]

4. Extensions

This paper has illustrated that when a monopolist faces a continuum of consumers whose purchasing decisions evolve according to a broad generalization of the replicator dynamics, then the monopolist eventually exits. We now discuss the robustness of this result to modeling extensions.

First we note that our results would continue to hold even if the monopolist were selling to a continuum of firms rather than a continuum of consumers. If there is a continuum of firms

[10] While Theorem 1 guarantees that monopolies will eventually exit, the amount of time it takes to reach the limiting behavior in Theorem 1 depends significantly on the parameters of the game and can range from a few decades (for impatient firms) to several centuries (for patient firms), as we show in Section B of the appendix. The fact that it may take a long time for sales to become small may mean that the replicator dynamics provides a reasonable approximation for an empirical study over a short time frame, but it does not change the fact that it does not provide a satisfactory model of consumers over long time horizons.

with different values for a monopolist's product, and we model the evolution of these firms' purchasing decisions using the model in Section 2, then the same reasoning can be used to establish that the monopolist eventually stops selling to these firms. This suggests that it may also be inappropriate to model the evolution of firms using the replicator dynamics.

Next note that our results would also continue to hold if there were externalities and a consumer's willingness to pay depends on the fraction of other consumers that are purchasing the product. To incorporate externalities into the model, we could model a consumer's willingness to pay by $v(\theta, t, \overline{x}(t))$ so that a consumer's willingness to pay depends on $\overline{x}(t)$ in addition to θ and t. Under this alternative model, all of the results in our manuscript continue to hold because none of the proofs of our results in any way depend on the assumption that $v(\cdot)$ is independent of the fraction of other consumers that are purchasing the product.

Also note that the assumption that there is a continuum of consumer types is not crucial to derive the main substantive result of the paper. In a model with bounded but discrete types, nearly identical reasoning can be used to establish that, if the maximum price is large enough, only consumers with the highest type will consume the product after a long time has passed. Thus the monopolist's sales eventually become very small even if there is not a continuum of types.

Finally, we address the question of whether the result in Theorem 1 will continue to hold if one weakens the assumption that the rate of change in the quantity purchased goes to zero in the limit as the fraction of consumers purchasing the product tends to zero or one. In our main model, we have focused on a setting in which $\lim_{x \to 0} g(x) = \lim_{x \to 1} g(x) = 0$, so the rate of change in the quantity purchased goes to zero in the limit as the fraction of consumers purchasing the product tends to zero or one. However, one could alternatively model evolution by assuming that $g(x)$ is bounded away from zero for all $x \in [0, 1]$, and then assume that $\dot{x}(\theta, t) = \max\{0, f(v(\theta, t) - p(t))g(x(\theta, t))h(\bar{x}(t))\}$ when $x(\theta, t) = 0$ and $\dot{x}(\theta, t) = \min\{0, f(v(\theta, t) - p(t))g(x(\theta, t))h(\bar{x}(t))\}$ when $x(\theta, t) = 1$.

Under this alternative model, the rate of change in the quantity purchased does not go to zero in the limit as the fraction of consumers purchasing the product tends to zero or one. However, the fact that $\dot{x}(\theta,t) \geq 0$ when $x(\theta,t) = 0$ and $\dot{x}(\theta,t) \leq 0$ when $x(\theta,t) = 1$ ensures that the fraction of consumers of a given type that purchase the product will never fall below zero or rise above one. Under this setting, we can prove that as long as there is positive measure of consumer types whose values for the product exceed the monopolist's costs, the monopolist will not exit:

Theorem 2. *Suppose there is some $\theta^* < \overline{\theta}$ such that $v(\theta^*) \equiv \liminf_{t\to\infty} v(\theta^*,t) > c$. Then under the alternative model of consumer evolution in which $g(x)$ is bounded away from zero for all $x \in [0,1]$, the fraction of consumers who purchase the product does not tend towards zero in the limit as $t \to \infty$.*

Theorem 2 suggests that the key problem with the replicator dynamics that results in the unrealistic behavior in Theorem 1 is the assumption that consumers evolve so slowly near the extremes of zero and unit market shares. In general, we would expect consumers to evolve more quickly near these extremes than the replicator dynamics predicts since the replicator dynamics implies these consumers are very slow to change purchasing decisions in these cases, even if the monopolist is charging a very low or very high price.

5. Conclusion

This paper has illustrated that when a monopolist faces a continuum of consumers whose purchasing decisions evolve according to a broad generalization of the replicator dynamics, then the monopolist eventually exits. The reason for this finding is that if a significant fraction of consumers are still purchasing the product after a long time has passed, then evolution implies that nearly all consumers with values greater than the marginal type will be purchasing the product. The replicator dynamics implies that almost all of these consumers will continue to purchase the product for a very long time even if the monopolist charges these consumers more than their value. This in turn creates an incentive for the monopolist to raise prices, which eventually causes all consumers to stop purchasing the product.

The conclusion that monopolies exit when faced with evolutionary consumers seems at odds with reality and suggests that the evolutionary model of consumers based on the replicator dynamics may not accurately reflect the evolution of consumer strategies. While this does not establish what the best theory would be, there are several possible alternatives where consumers may be more sophisticated than they are in an evolutionary model based on the replicator dynamics.

First, under the replicator dynamics, consumers only change strategies at a rate proportional to the instantaneous difference in utility obtained as a result of adopting or dropping the product. However, if a consumer is currently being charged more than the amount he values the product, that consumer may anticipate that this overcharging will continue and thus exit more rapidly than the replicator dynamics would predict. The replicator dynamics also tacitly assumes history independence. However, consistent patterns may be easier to detect and act on, so a firm that is consistently charging more than a consumer's value may spawn an outsized reaction.

As a whole, a theory that allows consumers to evolve more quickly near the extremes of zero and unit market shares would seem to better describe reality, as such an alternative model of evolution would not imply that monopolies exit. The replicator dynamics may be plausible in economic situations where the extreme of extinction of a strategy is unlikely, but alternative models (such as best response dynamics considered in Dindoš and Mezzetti (2006)) are likely needed to describe evolution once nearly all players have adopted a particular strategy.

Appendix A. Proofs of Main Results

Lemma 0. *$\int_0^{x_0} \frac{1}{g(x)}\, dx = \int_{x_0}^1 \frac{1}{g(x)}\, dx = \infty$ for all $x_0 \in (0,1)$.*

Proof. Since $g(x)$ is a Lipschitz continuous function on $[0,1]$ satisfying $g(0) = 0$, there exists some $\beta > 0$ for which $g(x) \leq \beta x$ for all x, which in turn implies that $\frac{1}{g(x)} \geq \frac{1}{\beta x}$ for all x. Thus $\int_0^{x_0} \frac{1}{g(x)}\, dx \geq \int_0^{x_0} \frac{1}{\beta x}\, dx = \infty$, so $\int_0^{x_0} \frac{1}{g(x)}\, dx = \infty$ for all $x_0 \in (0,1)$. A similar argument shows that $\int_{x_0}^1 \frac{1}{g(x)}\, dx = \infty$ for all $x_0 \in (0,1)$. □

Proof of Lemma 1: Since $\frac{dx(\theta,t)}{dt} = f(v(\theta,t)-p(t))g(x(\theta,t))h(\bar{x}(t))$, it follows that $\frac{dx(\theta,t)}{g(x(\theta,t))} = f(v(\theta,t)-p(t))h(\bar{x}(t))\ dt$, meaning $\int_{x(\theta,0)}^{x(\theta,T)} \frac{1}{g(x)}\ dx = \int_0^T f(v(\theta,t)-p(t))h(\bar{x}(t))\ dt$. Thus if $G(x)$ is an anti-derivative of $\frac{1}{g(x)}$, then it must also be the case that $G(x(\theta,T)) - G(x(\theta,0)) = \int_0^T f(v(\theta,t)-p(t))h(\bar{x}(t))\ dt$, meaning $G(x(\theta,T)) = G(x(\theta,0)) + \int_0^T f(v(\theta,t)-p(t))h(\bar{x}(t))\ dt$.

Now since $G(x)$ is an anti-derivative of $\frac{1}{g(x)}$ and $\frac{1}{g(x)}$ is strictly positive for all $x \in (0,1)$, it must be the case that $G(x)$ is a strictly increasing function in x, so the inverse of $G(x)$, $G^{-1}(x)$, is well-defined. By using this insight and the result in the previous paragraph, it then follows that $x(\theta,T) = G^{-1}[G(x(\theta,0)) + \int_0^T f(v(\theta,t)-p(t))h(\bar{x}(t))\ dt]$. Thus if $\phi(x) \equiv G^{-1}(x)$, then it follows that $x(\theta,T) = \phi(\phi^{-1}(x(\theta,0)) + \int_0^T f(v(\theta,t)-p(t))h(\bar{x}(t))\ dt)$.

To prove the result, it thus suffices to show that $\phi(x) = G^{-1}(x)$ is a continuous and strictly increasing function satisfying $\lim_{x\to-\infty}\phi(x) = 0$ and $\lim_{x\to\infty}\phi(x) = 1$. To see this, note that since $G(x)$ is a strictly increasing function in x, it must also be the case that $\phi(x) = G^{-1}(x)$ is a strictly increasing function in x. And since $G(x)$ is an anti-derivative of a continuous function on $(0,1)$, it follows that $\phi(x) = G^{-1}(x)$ must also be continuous on $(0,1)$. Finally since $\int_0^{x_0} \frac{1}{g(x)}\ dx = \int_{x_0}^1 \frac{1}{g(x)}\ dx = \infty$ for all $x_0 \in (0,1)$ (by Lemma 0) and $G(x)$ is an anti-derivative of $\frac{1}{g(x)}$, it must be the case that $\lim_{x\to 0} G(x) = -\infty$ and $\lim_{x\to 1} G(x) = \infty$. From this it follows that $\phi(x) = G^{-1}(x)$ satisfies $\lim_{x\to-\infty}\phi(x) = 0$ and $\lim_{x\to\infty}\phi(x) = 1$. □

Throughout the remainder of the appendix we make use of the following definition:

Definition 1. *Define* $T(\epsilon)$ *to be the value of* $\frac{\log(\epsilon)}{\log(\delta)}$.

Proof of Lemma 2: Let $f_\Delta(\epsilon)$ denote the smallest value of $f(v) - f(v-\epsilon)$ taken over all values of $v \in [\underline{v} - P, \overline{v}]$ (where $\underline{v} \equiv \inf_t v(\underline{\theta}, t)$), and let $\underline{h}$ denote the smallest value of $h(x)$ for all $x \in [0,1]$. Also let H denote the largest value of $h(x)$ for all $x \in [0,1]$. Finally let ϕ_Δ denote the largest value of the absolute difference $|\phi^{-1}(x(\theta,0)) - \phi^{-1}(x(\theta',0))|$ for all values of θ and θ' in $[\underline{\theta}, \overline{\theta}]$.

Note that decreasing the value of θ by at least ϵ decreases the value of the integral $\int_0^{t^*} f(v(\theta,t)-p(t))h(\bar{x}(t))\ dt$ by a minimum of $t^* f_\Delta(\tilde{v}(\epsilon))\underline{h}$. Thus decreasing the value of θ

by at least ϵ also decreases the value of $\phi^{-1}(x(\theta,0)) + \int_0^{t^*} f(v(\theta,t) - p(t))h(\bar{x}(t))\, dt$ by a minimum of $t^* f_\Delta(\tilde{v}(\epsilon))\underline{h} - \phi_\Delta$. At the same time, the value of $\int_0^{t^*+T(\epsilon)} f(v(\theta,t) - p(t))h(\bar{x}(t))\, dt$ can never exceed the value of $\int_0^{t^*} f(v(\theta,t) - p(t))h(\bar{x}(t))\, dt$ by more than $f(\overline{v})HT(\epsilon)$.

Thus if T satisfies $Tf_\Delta(\tilde{v}(\epsilon))\underline{h} > \phi^{-1}(1-\epsilon) - \phi^{-1}(\epsilon) + \phi_\Delta + f(\overline{v})HT(\epsilon)$ and $x(\theta^*, t^*) = y$ for some $t^* \geq T$ and $y \in (\epsilon, 1-\epsilon)$, then it is necessarily the case that $x(\theta,t) < \epsilon$ for all $\theta \leq \theta^* - \epsilon$ and $t \in [t^*, t^* + T(\epsilon)]$. A similar argument shows that if T satisfies $Tf_\Delta(\tilde{v}(\epsilon))\underline{h} > \phi^{-1}(1-\epsilon) - \phi^{-1}(\epsilon) + \phi_\Delta + |f(\underline{v} - P)|HT(\epsilon)$ and $x(\theta^*, t^*) \geq y$ for some $t^* \geq T$ and $y \in (\epsilon, 1-\epsilon)$, then it is necessarily the case that $x(\theta,t) > 1-\epsilon$ for all $\theta \geq \theta^* + \epsilon$ and $t \in [t^*, t^* + T(\epsilon)]$. □

Proof of Lemma 3: The proof breaks down into two cases. Suppose first that $x(\hat{\theta}, t^*) = y$ for some $\hat{\theta} \leq \theta^*$. If we consider the game beginning at the time t^*, then the firm's total payoff from using an arbitrary price path $p(t)$ from the game beginning at time t^* is $\int_{t^*}^{t^*+T(\epsilon)} \delta^{t-t^*}(p(t) - c)E_\theta[x(\theta,t^*)|t]\, dt + O(\epsilon)$, where $E_\theta[x(\theta,t^*)|t] \equiv \int_{\underline{\theta}}^{\overline{\theta}} x(\theta,t^*)\gamma(\theta,t)\, d\theta$: Note that $\delta^{T(\epsilon)} = e^{\log(\delta)T(\epsilon)} = e^{\log(\epsilon)} = \epsilon$, so the firm's total payoff from the part of the game for the time interval $t \geq t^* + T(\epsilon)$ is $O(\epsilon)$. And from Lemma 2, we know that $x(\theta,t) < \epsilon$ if $\theta \leq \hat{\theta} - \epsilon$ and $x(\theta,t) > 1-\epsilon$ if $\theta \geq \hat{\theta} + \epsilon$ for all $t \in [t^*, t^* + T(\epsilon)]$ regardless of the price path chosen by the firm. From this it follows that the total fraction of consumers who purchase the product at time t cannot differ from $E_\theta[x(\theta,t^*)|t]$ by more than $O(\epsilon)$ regardless of the price path chosen by the firm. Thus the firm's total payoff from using an arbitrary price path $p(t)$ from the game beginning at time t^* is $\int_{t^*}^{t^*+T(\epsilon)} \delta^{t-t^*}(p(t) - c)E_\theta[x(\theta,t^*)|t]\, dt + O(\epsilon)$. But this expression implies that a firm will maximize its payoff by using prices within $O(\epsilon)$ of the maximum possible price during the time interval $[t^*, t^* + T(\epsilon)]$. The result thus follows if $x(\hat{\theta}, t^*) = y$ for some $\hat{\theta} \leq \theta^*$.

If $x(\hat{\theta}, t^*) = y$ does not hold for any $\hat{\theta} \leq \theta^*$, then it must be the case that $x(\hat{\theta}, t^*) > y$ for all $\hat{\theta} \leq \theta^*$, and in particular that $x(\underline{\theta}, t^*) > y$. If we consider the game beginning at the time t^*, then the firm's total payoff from using an arbitrary price path $p(t)$ from the game beginning at time t^* is again $\int_{t^*}^{t^*+T(\epsilon)} \delta^{t-t^*}(p(t) - c)E_\theta[x(\theta,t^*)|t]\, dt + O(\epsilon)$: The firm's total payoff from the part of the game for the time interval $t \geq t^* + T(\epsilon)$ is again $O(\epsilon)$, and from Lemma 2, we know that $x(\theta,t) > 1-\epsilon$ if $\theta \geq \underline{\theta} + \epsilon$ for all $t \in [t^*, t^* + T(\epsilon)]$ regardless of the price path chosen by the firm. From this it follows that the total fraction of consumers who purchase the product cannot change by more than $O(\epsilon)$ during the time interval $[t^*, t^*+T(\epsilon)]$ regardless of the price path chosen by the firm. Thus the firm's total payoff from using an arbitrary price path $p(t)$ from the game beginning at time t^* is $\int_{t^*}^{t^*+T(\epsilon)} \delta^{t-t^*}(p(t)-c)E_\theta[x(\theta,t^*)|t]\,dt + O(\epsilon)$. But this expression implies that a firm will maximize its payoff by using prices within $O(\epsilon)$ of the maximum possible price during the time interval $[t^*, t^*+T(\epsilon)]$. The result thus follows if $x(\hat{\theta}, t^*) = y$ does not hold for any $\hat{\theta} \leq \theta^*$, which proves the desired result. □

Proof of Theorem 1: Suppose by means of contradiction that this result does not hold. Then there exists some fraction $y > 0$ and some $\theta^* \in (\underline{\theta}, \overline{\theta})$ such that there is no value of T for which $t > T$ implies $x(\theta, t) < y$ if $\theta \leq \theta^*$. For any given value of $\theta^* \in (\underline{\theta}, \overline{\theta})$, let $Y(\theta^*)$ denote the set of all values of $y > 0$ for which there is no value of T such that $t > T$ implies $x(\theta, t) < y$ if $\theta \leq \theta^*$, and let $y(\theta^*)$ denote the supremum of $Y(\theta^*)$. Now consider two cases.

First suppose there is some value of $\theta^* \in (\underline{\theta}, \overline{\theta})$ for which $y(\theta^*) \in (0, 1)$. Consider some arbitrarily small value of $\epsilon > 0$, and let $T_d(\epsilon)$ denote a value of T corresponding to this value of ϵ that satisfies the condition given in Lemma 2. Note that there must be some time $t^* \geq T_d(\epsilon)$ for which $x(\theta^*, t^*) \in [y(\theta^*)-\epsilon, y(\theta^*)+\epsilon]$ because if this did not hold then it would either be the case that $x(\theta^*, t) > y(\theta^*) + \epsilon$ for all $t \geq T_d(\epsilon)$ or it would be the case that $x(\theta^*, t) < y(\theta^*) - \epsilon$ for all $t \geq T_d(\epsilon)$, either of which would contradict the definition of $y(\theta^*)$. Thus there is some time $t^* \geq T_d(\epsilon)$ for which $x(\theta^*, t^*) \in [y(\theta^*) - \epsilon, y(\theta^*) + \epsilon]$.

But for any such time t^*, we know from Lemma 3 that a firm will maximize its payoff by using prices within $O(\epsilon)$ of the maximum possible price during the time interval $[t^*, t^*+T(\epsilon)]$. Thus if $x(\theta^*, t^*) \in [y(\theta^*)-\epsilon, y(\theta^*)+\epsilon]$, then it will necessarily be the case that $x(\theta, t^*+T(\epsilon))$ is much less than $y(\theta^*) - \epsilon$ for all values of $\theta \leq \theta^*$. And similar reasoning shows that under the optimal price path, $x(\theta, t)$ can never become anywhere near as large as $y(\theta^*) - \epsilon$ for any values of $t \geq t^* + T(\epsilon)$ if $\theta \leq \theta^*$. This contradicts the definition of $y(\theta^*)$ and proves that this case cannot hold.

Next suppose there is no value of $\theta^* \in (\underline{\theta}, \overline{\theta})$ for which $y(\theta^*) \in (0, 1)$. Since we know that there is some value of $\theta^* \in (\underline{\theta}, \overline{\theta})$ for which $y(\theta^*) > 0$, this in turn implies that there is some value of $\theta^* \in (\underline{\theta}, \overline{\theta})$ for which $y(\theta^*) = 1$. Moreover, by the definition of $y(\theta)$, $y(\theta)$ must be non-decreasing in θ. From this it follows that there must exist some $\hat{\theta} \in [\underline{\theta}, \overline{\theta})$ for which $y(\theta) = 1$ if $\theta > \hat{\theta}$ and $y(\theta) = 0$ if $\theta < \hat{\theta}$.

Now consider some type $\tilde{\theta} \in (\hat{\theta}, \overline{\theta})$. Since $y(\tilde{\theta}) = 1$, it follows that for any $y \in (0, 1)$ and any time T there exists some $t^* > T$ for which $x(\tilde{\theta}, t^*) > y$. Thus for any arbitrarily small $\epsilon > 0$, if $t^* > T_d(\epsilon)$, then we know from Lemma 3 that a firm will maximize its payoff by using prices within $O(\epsilon)$ of the maximum possible price during the time interval $[t^*, t^* + T(\epsilon)]$. But since this held for any $t^* > T_d(\epsilon)$ for which $x(\tilde{\theta}, t^*) > y$, it then follows that $x(\tilde{\theta}, t)$ remains bounded above by $x(\tilde{\theta}, t^*)$ for all $t \geq t^*$, and thus $x(\theta, t)$ remains bounded away from 1 for all $\theta \leq \tilde{\theta}$ and $t \geq t^*$. Since this contradicts the fact that $y(\tilde{\theta}) = 1$, this proves that this case cannot hold. The result then follows. □

Proof of Theorem 2: Suppose by means of contradiction that the fraction of consumers who purchase the product tends towards zero in the limit as $t \to \infty$. In that case, for any $\epsilon > 0$, there exists some T such that the fraction of consumers purchasing the product is less than ϵ for all $t \geq T$, which in turn implies that the monopolist obtains a payoff no greater than $\frac{\epsilon(P-c)}{1-\delta}$ for the game starting from time $t = T$. However, if the monopolist instead offers to sell the products for a price $p(t) = \frac{v(\theta^*)+c}{2}$ for all $t \geq T$, then since $v(\theta, t) - p(t) \geq \frac{v(\theta^*)-c}{2} > 0$ for all $\theta \geq \theta^*$ and $t \geq T$, and $g(x)$ is bounded away from zero for all x, it follows that there is some $d > 0$ such that $\dot{x}(\theta, t) > d$ for all $\theta \geq \theta^*$ and $t \geq T$ as long as $x(\theta, t) < 1$.

The above result implies that if the monopolist instead offers to sell the products for a price $p(t) = \frac{v(\theta^*)+c}{2}$ for all $t \geq T$, then after $\Theta(\epsilon^{1/3})$ time has passed, the fraction of consumers of type $\theta \geq \theta^*$ who will be purchasing the product will be $\Omega(\epsilon^{1/3})$. This in turn implies that the firm obtains a profit of $\Omega(\epsilon^{1/3})$ at any given point in time once $\Theta(\epsilon^{1/3})$ time has passed, which in turn implies that the firm obtains an overall profit of $\Omega(\epsilon^{1/3})$ for the game starting from time $t = T$ by following this strategy. This then means the monopolist can obtain a greater profit starting from time $t = T$ by following this strategy than by following a strategy in which the fraction of consumers purchasing the product is less than ϵ for all $t \geq T$.

Since the monopolist would be able to profitably deviate if the monopolist is following a pricing strategy such that the fraction of consumers who purchase the product tends towards zero in the limit as $t \to \infty$, it follows that the fraction of consumers who purchase the product does not tend towards zero in the limit as $t \to \infty$. □

Appendix B. Simulations

This section illustrates that the amount of time it will take to reach the limiting behavior in Theorem 1 will depend significantly on the parameters of the game. To illustrate this, we consider a setting in which consumer evolution is governed by the replicator dynamics in that if $x(v,t)$ denotes the fraction of consumers with value v who are consuming the product at time t and $p(t)$ denotes the price at time t, then $\dot{x}(v,t) = \lambda(v - p(t))x(v,t)(1 - x(v,t))$. We have seen in equation (2) that in this setting, we have

$$\frac{x(v,t)}{1 - x(v,t)} = \frac{x(v,0)}{1 - x(v,0)} e^{\lambda(vt - P(t))},$$

where $P(t) \equiv \int_0^t p(s)\, ds$. Thus we also have

$$x(v,t) = \frac{x(v,0)e^{\lambda(vt-P(t))}}{1 - x(v,0) + x(v,0)e^{\lambda(vt-P(t))}} = \frac{x(v,0)e^{\lambda vt}}{(1 - x(v,0))e^{\lambda P(t)} + x(v,0)e^{\lambda vt}}.$$

Now consider a setting in which the fraction of consumers with value v who are initially purchasing the product, $x(v,0)$, is some constant independent of v, and let $z \equiv \frac{x(v,0)}{1-x(v,0)}$. Also suppose that consumer values v are uniformly distributed on the interval $[0,1]$. Then total sales at time t are

$$\begin{aligned}
\int_0^1 x(v,t)\, dv &= \int_0^1 \frac{ze^{\lambda vt}}{e^{\lambda P(t)} + ze^{\lambda vt}}\, dv \\
&= \frac{1}{\lambda t}\int_0^1 \frac{z\lambda t e^{\lambda vt}}{e^{\lambda P(t)} + ze^{\lambda vt}}\, dv \\
&= \frac{1}{\lambda t}\log(e^{\lambda P(t)} + ze^{\lambda vt})\Big|_0^1 \\
&= \frac{1}{\lambda t}(\log(e^{\lambda P(t)} + ze^{\lambda t}) - \log(e^{\lambda P(t)} + z)), \qquad (3)
\end{aligned}$$

so the monopolist's total profit from using the pricing strategy $P(t)$ if the monopolist has zero costs and a discount factor of $\delta = e^{-r}$ is

$$\Pi = \int_0^\infty e^{-rt}\dot{P}(t)\frac{1}{\lambda t}(\log(e^{\lambda P(t)} + ze^{\lambda t}) - \log(e^{\lambda P(t)} + z))\, dt.$$

By applying the Euler-Lagrange equation, we then see that if $g(P(t), t)$ is defined by $g(P(t), t) \equiv e^{-rt}\frac{1}{\lambda t}(\log(e^{\lambda P(t)} + ze^{\lambda t}) - \log(e^{\lambda P(t)} + z))$ and $F(\dot{P}(t), P(t), t) \equiv \dot{P}(t)g(P(t), t)$, then the optimal $P(t)$ satisfies $\frac{\partial F}{\partial P} = \frac{d}{dt}\frac{\partial F}{\partial \dot{P}}$, which implies $\dot{P}(t)\frac{\partial g}{\partial P} = \frac{d}{dt}g(P(t), t) = \dot{P}(t)\frac{\partial g}{\partial P} + \frac{\partial g}{\partial t}$, which in turn implies $\frac{\partial g}{\partial t} = 0$. Thus the optimal $P(t)$ satisfies $\frac{\partial g}{\partial t} = 0$, meaning we have

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\left[e^{-rt}\frac{1}{\lambda t}(\log(e^{\lambda P(t)} + ze^{\lambda t}) - \log(e^{\lambda P(t)} + z))\right] \\
&= -\left(r + \frac{1}{t}\right)\left[e^{-rt}\frac{1}{\lambda t}(\log(e^{\lambda P(t)} + ze^{\lambda t}) - \log(e^{\lambda P(t)} + z))\right] + e^{-rt}\frac{1}{t}\frac{ze^{\lambda t}}{e^{\lambda P(t)} + ze^{\lambda t}} \\
&= e^{-rt}\frac{1}{t}\left[\frac{ze^{\lambda t}}{e^{\lambda P(t)} + ze^{\lambda t}} - \left(r + \frac{1}{t}\right)\frac{1}{\lambda}(\log(e^{\lambda P(t)} + ze^{\lambda t}) - \log(e^{\lambda P(t)} + z))\right],
\end{aligned}$$

which is in turn satisfied if and only if

$$\frac{z\lambda e^{\lambda t}}{e^{\lambda P(t)} + ze^{\lambda t}} - \left(r + \frac{1}{t}\right)(\log(e^{\lambda P(t)} + ze^{\lambda t}) - \log(e^{\lambda P(t)} + z)) = 0. \tag{4}$$

By using equation (4) to compute the optimal $P(t)$, we can then use the resulting $P(t)$ to compute sales at any given point in time using the expression for sales in equation (3).

By conducting such analysis, we find that the length of time it takes to reach the limiting behavior in Theorem 1 depends significantly on the parameters of the game. Suppose, for instance, that $\lambda = 1$ and $z = 100$ so that over 99% of consumers are initially purchasing the product. Then if firms are quite patient (*e.g.* $r = 0.02$), then firms retain a 10% market share even after 450 years. By contrast, if firms are impatient (*e.g.* $r = 0.2$), then the firm's market share drops below 10% after just 45 years. Thus the length of time it takes to reach the limiting behavior in Theorem 1 depends significantly on the parameters of the game.

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