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Multidimensional Incentive Compatibility and Mechanism Design

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Incentive-compatibility conditions are characterized in general for problems in which several goods are to be allocated and agents' types are multidimensional. Two questions of monopoly pricing under multidimensional uncertainty are then analyzed. First, we find the optimal nonlinear pricing scheme when the monopolist knows only the distribution of an arbitrarily parameterized family of demand curves, and second, we solve for the optimal bundling policy for a multiproduct monopolist. *Journal of Economic Literature* Classification Numbers: 022, 026.

1. INTRODUCTION

One of the most useful technical advances in economic theory in the last decade has been the development of the Revelation Principle. In itself a simple enough observation, the Revelation Principle states that any given recource-allocation process can be mimicked by a mechanism in which the participants are asked simply to reveal their private information, with the payoffs designed so that each individual finds it in his or her interest to tell the truth. The Revelation Principle has made possible the solution of a wide range of problems in which relevant information is dispersed among several individuals. The value of the Revelation Principle is that it reduces

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While the programming problem that results from applying the Revelation Principle is easy to state, it is not necessarily easy to solve. In particular, the incentive-compatibility constraints are often difficult to handle. For the sake of tractability, most existing applications of the Revelation Principle have restricted the dimensionality of the problem. Thus they have considered situations where only one or two goods are to be allocated and where any individual's private information can be summarized by a single number. These dimensionality assumptions mean that many interesting economic questions cannot be addressed.

Consider, for example, the pricing decision of a monopolist. According to elementary textbooks, the monopolist does best by perfectly price-discriminating, charging a price schedule that coincides with the buyer's demand curve. However, this presumes the monopolist has precise information about each customer's willingness to pay. Suppose, more realistically, that the monopolist cannot observe individual demands, but instead only knows the distribution of demands in the population. To what extent can he now achieve price discrimination?[2] For the uncertainty to be one-dimensional in this case, the monopolist must know the entire shape of any buyer's demand curve except, say, the intercept. Although the informational asymmetry is the crucial feature of this problem, the restriction to scalar-valued uncertainty assumes away most of the potential asymmetry of information.

Another monopoly problem with multidimensional uncertainty arises when the monopolist sells several different goods but does not know any one buyer's willingness to pay for each good. Stigler [23] and Adams and Yellen [1] showed by examples that the monopolist could extract surplus by bundling his products. Is bundling the expected-profit-maximizing selling policy?[3]

[1] For useful expositions of the Revelation Principle, see Harris and Townsend [8] and Myerson [18].

[2] The monopolist's problem with one-dimensional uncertainty was solved by Maskin and Riley [12], and a two-dimensional monopoly problem was solved by Laffont, Maskin, and Rochet [9].

[3] This problem was addressed by Palfrey [19]. Related questions have been modelled by Spence [22], Chiang and Spatt [3], Rochet [21], and McAfee, McMillan, and Whinston [15].

How misleading is the restriction to single-dimensional uncertainty? The results to follow can be interpreted as showing that, under certain conditions, multidimensional problems have solutions that resemble their single-dimensional counterparts. Thus the lessons derived from the single-dimensional analyses can be extrapolated to the much more complicated multidimensional models.

In what follows we present a general result as well as an example for the case in which the optimal mechanism is differentiable, and an example, introducing some new methods involving the Stokes theorem, illustrating the optimization of nondifferentiable mechanisms. In Section 2, we characterize multidimensional incentive compatibility in general. In Section 3, we generalize the analysis of Laffont, Maskin, and Rochet [9], in which the monopolist knows that each buyer's demand curve is linear but does not know its slope or intercept. In Section 4, we find the optimal mechanism in the problem of Palfrey [19], in which the monopolist sells several different goods and may choose to bundle them: the monopolist knows that his customers have linear utility functions but does not know their parameters. For the case of two goods, we provide a condition under which the optimal policy is nonstochastic; this condition also ensures it is in the monopolist's interest to bundle his outputs.

2. Characterization of Incentive Compatibility

Each individual is described by his type, which is a vector $t = (t_1, ..., t_k) \in R^k$. Types occur in the population according to a distribution $F(t)$, which is assumed to have a continuously differentiable density $f(t)$ possessing a support on a convex open region $D \subseteq R^k$.

A vector of commodities, $(x, y) = (x_1, ..., x_n, y) \in R^{n+1}$, is to be allocated. Each agent has a twice continuously differentiable utility function $U(x, y, t)$. The good y will be taken to be the numeraire; we assume therefore that $U_y > 0$ (subscripts denote partial derivatives).

Define an n-vector of shadow prices by

$$p(x, y, t) = \frac{U_x(x, y, t)}{U_y(x, y, t)}. \tag{1}$$

Two assumptions are maintained throughout the paper. First, there are at least as many types as goods, so that $n \leqslant k$. Second, the matrix p_t has full rank n (where a typical element in p_t is $\partial p_i / \partial t_j$). An example with $n > k$ is analyzed in Matthews and Moore [13].

A *mechanism* is a process that takes as inputs the agents' reports and produces as its output a decision about the allocation of resources.

A mechanism is *direct* if each agent is asked simply to report his private information. We examine a direct, deterministic, differentiable mechanism.[4] Such a mechanism is *incentive compatible* if it does not pay an individual to claim his type is something other than it really is; that is,

$$(\forall s, t \in D) \qquad U(x(s), y(s), t) \leqslant U(x(t), y(t), t). \tag{2}$$

Which direct, deterministic, continuously differentiable mechanisms satisfy incentive compatibility? We analyze first necessity and then sufficiency.

THEOREM 1. *If a direct, deterministic, differentiable mechanism is incentive compatible, then*

$$y'(t) = -p(x(t), y(t), t)\, x'(t) \tag{3}$$

and

$$x'(t) = Cp_t(x(t), y(t), t), \tag{4}$$

for some positive semidefinite $n \times n$ matrix C (which generally depends on t).

Proof. Let $V(s, t) = U(x(s), y(s), t)$. Incentive compatibility requires that V take a maximum at $s = t$, so that

$$V_s(t, t) = (0, 0, ..., 0) \tag{5}$$

and

$$V_{ss}(t, t) \text{ is negative semidefinite.} \tag{6}$$

Condition (5) implies (3), as required.

Condition (5) also implies that

$$V_{ss}(t, t) + V_{st}(t, t) = \mathbf{0} \tag{7}$$

(where $\mathbf{0}$ denotes the zero matrix). Now (6) and (7) imply $V_{st}(t, t)$ is positive semidefinite, or

$$U_{xt}x' + U_{yt}\, y' \text{ is positive semidefinite.} \tag{8}$$

Let a "T" superscript denote transpose. From (3),

$$U_y\, p_t^{\mathrm{T}} x' = (U_{xt} - U_{yt}\, p)x' \text{ is positive semidefinite.} \tag{9}$$

Thus, using $U_y > 0$,

$$p_t^{\mathrm{T}} x'(t) \text{ is positive semidefinite.} \tag{10}$$

[4] Section 4 will analyze an example of a nondifferentiable mechanism.

The Appendix proves the following lemma. Suppose A and B are $n \times k$ matrices, with $n \leqslant k$ and rank $(B) = n$. Then $B^{\mathrm{T}}A$ is positive semidefinite if and only if $A = CB$ for some $n \times n$ positive semidefinite matrix C. Applying this lemma to (10) and using the assumption that p_t has rank n yields (4), as required. Q.E.D.

In the one-good, one-type case ($k = n = 1$), (4) reduces to $x' = cp_t$ for some nonnegative number c, which says that, if the shadow price of x is increasing in t, then the higher an individual's t, the more x he should receive. If this were not the case, individuals would have clear incentives to misrepresent their types. This one-dimensional special case of Theorem 1 was obtained by Guesnerie and Laffont [7]; see also [6].

In one-dimensional problems, sufficiency is obtained by assuming the single-crossing property, which means that the shadow price of x is monotonic in type (see, for example, Cooper [4], Guesnerie and Laffont [7], and Maskin and Riley [12]). Our multidimensional analogue is as follows.

DEFINITION. The utility function U satisfies the generalized single-crossing property if for all s, t, x, y there exists $\lambda > 0$ such that

$$p(x, y, t) - p(x, y, s) = \lambda p_t(x, y, s)(t - s). \tag{11}$$

Note that, in this definition, λ may depend upon s, t, x, and y. In the one-dimensional case, $k = n = 1$, (11) reduces to the standard single-crossing property (hence the name generalized single crossing). In the case of a single good to be allocated ($n = 1$), generalized single crossing merely asserts that if the shadow price p is increasing in the direction $t - s$ locally, then it is increasing in this direction globally.

With arbitrary numbers of goods and types, a sufficient condition for generalized single crossing is that the price function p be linear in types. (Cf. Rochet [21].) Linearity is equivalent to $\lambda = 1$ globally.

Generalized single crossing causes the price vectors (as functions of types alone) to line up: that is, the same number λ works for all n prices. In other words, define λ by

$$\lambda = \frac{p_{1t}(x, y, s) \cdot (t - s)}{p_1(x, y, t) - p_1(x, y, s)},$$

where the "1" subscript in p_{1t} denotes the first row vector. Then generalized single crossing says, first, that $\lambda > 0$ and, second, that, for $n \geqslant j \geqslant 2$,

$$p_j(x, y, t) = p_j(x, y, s) + \lambda p_{jt}(x, y, s) \cdot (t - s).$$

This shows the sense in which prices line up: the scaling factor λ is the same for all prices.[5]

THEOREM 2. *In a direct, deterministic, differentiable mechanism, given that* $n \leqslant k$, p_t *has rank* n, *and the generalized-single-crossing property* (11) *holds, incentive compatibility is equivalent to* (3) *and* (4).

Proof. The definition of generalized single crossing, (11), implies that, for any positive semidefinite matrix C,

$$(p(x, y, t) - p(x, y, s))\, C p_t(x, y, t)(t-s) \geqslant 0 \tag{12}$$

and thus, by (4),

$$(p(x, y, t) - p(x, y, s))\, x'(t)(t-s) \geqslant 0. \tag{13}$$

Let $t = \alpha z + (1-\alpha)s$ so that $t - s = \alpha(z-s)$. Now

$$\begin{aligned} V(z, s) - V(s, s) &= \int_0^1 \frac{\partial}{\partial \alpha} V(\alpha z + (1-\alpha)s, s)\, d\alpha \\ &= \int_0^1 V_1(\alpha z + (1-\alpha)s, s) \cdot (z-s)\, d\alpha \\ &= \int_0^1 \frac{1}{\alpha} [U_x(x(t), y(t), s)\, x'(t) \\ &\quad + U_y(x(t), y(t), s)\, y'(t)](t-s)\, d\alpha \\ &= -\int_0^1 \frac{U_y}{\alpha} [p(x(t), y(t), t) \\ &\quad - p(x(t), y(t), s)]\, x'(t)(t-s)\, d\alpha \leqslant 0 \end{aligned} \tag{14}$$

(the last step using $U_y > 0$, (3), and (13)). Since this is nonpositive for any z, s, generalized single crossing (11) makes (3) and (4) sufficient. Q.E.D.

Theorems 1 and 2 characterize incentive compatibility for a wide range of problems. They do not, however, provide an automatic solution for any particular multidimensional asymmetric-information problem. They reduce the incentive-compatibility conditions to a vector partial differential equation, which itself may be difficult to solve. In the next section, we

[5] Note that, if U satisfies generalized single crossing, then so does $h(U)$ for any strictly motonic increasing function h. In addition, generalized single crossing, when combined with the rank condition, is equivalent to $p(x, y, t)$ = constant defining a hyperplane in type space. However, the hyperplanes need not be parallel, as the constant varies.

illustrate the application of Theorems 1 and 2 by solving a problem of monopoly pricing. As will be seen, the assumption that utility satisfies single crossing facilitates the solution of the partial differential equation.[6]

Theorems 1 and 2 restricted attention to mechanisms that are both deterministic and differentiable. In some applications, however, the optimal mechanism is either stochastic or nondifferentiable. In Section 4 we solve a multidimensional problem with a nondifferentiable optimal mechanism.[7]

It should be noted that the generalized single-crossing property is not as restrictive as it appears in (11) because the units of type space have no natural meaning. Thus, effectively, (11) must hold for some (invertible) transformation of the initial units t. If we let $\beta: R^k \to R^k$ be an invertible transformation, the generalized single-crossing property (GSCP) may be relaxed to require: $p(x, y, t) - p(x, y, s) = \lambda p_t(x, y, s)\, \beta'(s)^{-1}(\beta(t) - \beta(s))$, for some β. While this is a strictly weaker property, it does not appear to correspond to any natural assumption on the untransformed problem (with initial units t, we have new units $\beta(t)$ and initial shadow price vector $p(x, y, \beta^{-1}(t))$).

Generally, GSCP does not reduce to linearity. To see this, suppose for a given (x, y) that p can be transformed to being linear in type by some transformation $\beta(t)$ of the type variable (such transformations are, of course, irrelevant). Generally the same transformation will fail to linearize p for other (x, y) values: examples are easy to construct when $k = 2$.

[6] These theorems assume there is adverse selection but no moral hazard. If the principal obtains no extra information ex post, moral hazard can be added simply by presuming that the utility function U embodies the fact that each agent will choose his best action as a function of his type. If, however, the principal receives further information after the actions are chosen, then (as shown in McAfee and McMillan [14]) the moral hazard interacts with the adverse selection and the foregoing results do not apply. It will become clear, in Section 3, that the generalized single-crossing property makes multidimensional problems behave in a manner similar to single-dimensional problems. We thank David Sappington for this observation.

[7] The analysis of Rochet [21] can be contrasted with the present study in two ways. First, Rochet assumes linearity in types, which is sufficient, but not necessary, for generalized single crossing. Thus, our analysis admits more general utility functions than Rochet's. Second, Rochet's results amount to the envelope theorem for this problem. That is, Rochet shows (Theorem 1, p. 117), in our notation,

$$\frac{\partial}{\partial t} U(x, y, t \mid x = x(t), y = y(t)) = \frac{d}{dt} U(x(t), y(t), t), \qquad \text{a.e. } t.$$

The advantage of Rochet's analysis is that, for utility linear in type, incentive compatibility reduces to $Q(t) = U(x(t), y(t), t)$ being convex in t. This is clearly an implication of our Section 1 (for utility linear in t), but we need to assume differentiability, whereas Rochet does not. Our analysis in Section 3 mirrors Rochet's analysis (without differentiability) and provides an

3. Monopoly with Unknown Demands

To illustrate the application of Theorems 1 and 2, consider the pricing decision of a monopolist who knows that his customers have utility functions from a parameterized family of the form

$$U(x, y, t) = V(x, t) + y, \tag{15}$$

where $-y$ is payment, x is consumption $(n = 1)$, $t \in R^k$ parameterizes the utility function, and $V(0, t) = 0$. The demand function (shadow price of consumption) is then

$$p(x, t) = V_x(x, t). \tag{16}$$

We suppose that the density of types is f defined over the unit cube $D = \times_{i=1}^{k} [0, 1]$. The generalized single-crossing property (11) requires that

$$p(x, t) - p(x, s) = \lambda p_t(x, s) \cdot (t - s). \tag{17}$$

This forces isoprice surfaces in type space to be hyperplanes, since they are described, for fixed s, by

$$\{t : p_t(x, s) \cdot (t - s) = 0\}. \tag{18}$$

Moreover, from Theorem 1, both x and y are constant on these hyperplanes.[8] We suppose the seller's cost of production is zero, and the seller maximizes expected revenue. The problem is to derive the optimal mechanism $x(t)$, $y(t)$. A special case of this problem was solved by Laffont, Maskin, and Rochet [9], for $p(x, t_1, t_2) = t_1 - t_2 x$ and f uniform on $[0, 1] \times [\frac{1}{2}, 1]$.

From (17), p is monotonic in t_i, holding the other types constant. It is useful to assume

$$\frac{\partial}{\partial t_1} p(x, t) > 0. \tag{19}$$

Let

$$t_{-1} = (t_2, ..., t_k). \tag{20}$$

[8] The simple way to see this is to note there is a nonlinear "price" $y(x)$ (y is the income remaining after paying the price for quantity x). Consumers of type s maximize $v(x, s) + y(x)$, yielding

$$0 = p(x, s) + y'(x).$$

Thus, $x =$ constant isoquantity lines occur on sets $\{t / p(x, s) = p(x, t)\}$, which, by (17), are of the form (18).

We shall also assume

$$(\forall t_{-1} \geqslant 0)(\forall x \geqslant 0) \qquad V(x, 0, t_{-1}) \leqslant 0. \tag{21}$$

Individual rationality guarantees $y(0, t_{-1}) \geqslant 0$ for all t_{-1}. Individual rationality requires the buyer to expect nonnegative utility from participation.

Let s stand for a vector that is zero in every position save the first:

$$s = (s_1, 0, ..., 0). \tag{22}$$

Then, by Theorem 1, we have that

$$p_t(x(s), s) \cdot (t - s) = 0 \text{ implies } x(t) = x(s) \text{ and } y(t) = y(s). \tag{23}$$

Thus, letting $z(s_1) = x(s)$,

$$y\left(s_1 - \sum_{i=2}^{k} t_i \frac{p_{t_i}(z(s_1), s)}{p_{t_1}(z(s_1), s)}, t_2, ..., t_n\right) = y(s), \tag{24}$$

which is obtained by solving (23) for t_1, in light of (22). From (3),

$$\begin{aligned} \frac{\partial}{\partial s_1} y(s) &= -p(x(s), s) \frac{\partial}{\partial s_1} x(s) \\ &= -p(z(s_1), s)\, z'(s_1). \end{aligned} \tag{25}$$

Furthermore,

$$z'(s_1) = \frac{\partial}{\partial t_1} x(t_1, 0, ..., 0) = c p_{t_1}(z(s), s) \geqslant 0 \tag{26}$$

by (4) and (19).

The following notation simplifies the problem:

$$F(t_1, ..., t_k) = \int_0^{t_1} f(\alpha, t_2, ..., t_k)\, d\alpha \tag{27}$$

$$h(s_1, t_{-1}) = s_1 - \sum_{i=2}^{k} t_i \frac{p_{t_i}(z(s_1), s)}{p_{t_1}(z(s_1), s)}. \tag{28}$$

We shall suppress the arguments of h, for clarity. Equation (28) provides a natural change of variables, which we call the LMR change of variables, since Laffont, Maskin, and Rochet [9] discovered it for their special case. Its significance arises from (23), for, letting t_{-1} vary, x and y are constant along $(h(s_1, t_{-1}), t_{-1})$, that is, h describes isoprofit surfaces in the type space. The effect of the generalized single-crossing property is to make these isoprofit surfaces hyperplanes.

The monopolist's profits are

$$\begin{aligned}
\Phi &= -\int_D y(t)\, f(t)\, dt \\
&= -\int_0^1 \cdots \int_0^1 y(t)\, f(t)\, dt_1 \cdots dt_k \\
&= -\int_0^\infty \cdots \int_0^\infty y(t)\, f(t)\, dt_1 \cdots dt_k \qquad \text{(LMR change of variables)} \\
&= -\int_0^\infty \cdots \int_0^\infty y(h, t_{-1})\, f(h, t_{-1}) \frac{\partial h}{\partial s_1}\, ds_1\, dt_{-1} \\
&= \int_0^\infty \cdots \int_0^\infty y(h, t_{-1}) \left[\frac{\partial}{\partial s_1} F(\infty, t_{-1}) - F(h, t_{-1}) \right] ds_1\, dt_{-1} \\
&= -\int_0^\infty \cdots \int_0^\infty y(h, t_{-1})[F(\infty, t_{-1}) - F(h, t_{-1})]|_{s_1=0}\, dt_{-1} \\
&\quad + \int_0^\infty \cdots \int_0^\infty \left[\frac{\partial}{\partial s_1} y(h, t_{-1}) \right] [F(\infty, t_{-1}) - F(h, t_{-1})]\, ds_1\, dt_{-1} \\
&\leqslant \int_0^\infty \cdots \int_0^\infty p(z(s_1), s)\, z'(s_1)[F(\infty, t_{-1}) - F(h, t_{-1})]\, ds_1\, dt_{-1} \\
&\qquad \text{(by (20), (24), and (25))} \\
&= \int_0^\infty p(z(s_1), s)\, z'(s_1) \left[1 - \int F(h, t_{-1})\, dt_{-1} \right] ds_1 \\
&\qquad \text{(by Fubini's theorem).} \qquad (29)
\end{aligned}$$

This is an ordinary calculus of variations problem in $z(s_1)$, $z'(s_1)$, and s_1, with the special form that it is multiplicative in $z'(s_1)$. The Euler equation, then, has the form[9]

$$\frac{\partial}{\partial s_1} p(z(s_1), s_1, 0, ..., 0) \left[1 - \int_0^\infty \cdots \int_0^\infty F(h, t_{-1})\, dt_{-1} \right] = 0. \qquad (30)$$

[9] To see this, note that if $F(x, x', t) = \alpha(x, t)x'$,

$$\alpha_x x' = \frac{\partial F}{\partial x} = \frac{d}{dt} \frac{\partial F}{\partial x'} = \alpha_x x' + \alpha_t$$

or $\partial \alpha / \partial t = 0$.

Suppose $\hat{z}(s_1)$ solves this equation, and

$$z^*(s_1) = \max\{0, \hat{z}(s_1)\}. \tag{31}$$

If z^* is nondecreasing (satisfying (26)) then z^* gives the solution to the problem, in conjunction with

$$x(h, t_{-1}) = z^*(s_1). \tag{32}$$

The value $y(h, t_{-1})$ is recovered by integrating (25) with

$$y(h, t_{-1})|_{s_1=0} = 0. \tag{33}$$

Since there is a free end-value for z, it follows that the type with the highest value of s_1 has a zero shadow price of further consumption, by transversality.

One significant special case occurs when we use polynomial approximations

$$p(x, t) = \sum_{i=1}^{k} t_i x^{i-1}, \tag{34}$$

which can satisfy all assumptions made. Then (30) becomes

$$\frac{\partial}{\partial s_1} s_1 \left[1 - \int_0^\infty \cdots \int_0^\infty F\left(s_1 - \sum_{i=2}^{n} t_i z(s_1)^{i-1}, t_2, ..., t_k \right) dt_{-1} \right] = 0,$$

which is in the form (for distribution G)

$$s_1 - \frac{1 - G(s_1, z)}{g(s_1, z)} = 0;$$

that is, it is a generalization of Maskin and Riley's Example 1 [12, p. 183] (given that production cost is zero). Indeed, the general version embodied in Eq. (30) shows that, for an adjusted distribution, the multidimensional optimization corresponds to the single-dimensional case. (This similarity of the multidimensional solution to the well-known single-dimensional solution is obscured in the Laffont–Maskin–Rochet [9] analysis because of their use of special functional forms.) As in the one-dimensional case, the individual with the highest possible type gets the efficient quantity of the good, and individuals with very low types get nothing. The seller acts like a perfectly discriminating monopolist with respect to a demand curve that is pushed inward relative to the true demand curve. For the utility function $U(x, y, t) = tx - \alpha x^2/2 - y$, where α is common knowledge so that utility is one-dimensional, the optimum satisfies $x(t) = \max\{0, 1 - \alpha[t - (1 - F)/f]\}$, which is consistent with Maskin and Riley [12]. In this way, the qualitative results of the one-dimensional case generalize to the n-dimensional case.

4. Multiproduct Monopoly

Consider a monopolist selling n indivisible commodities. Each buyer desires at most one unit of each commodity; a buyer's utility is given by

$$U(x, y, t) = \sum_{i=1}^{n} t_i x_i - y = t \cdot x - y, \tag{35}$$

where $x_i \in [0, 1]$ is the probability the buyer receives good i. The density of t, $f(t)$, is defined on $D = \times_{i=1}^{n} [0, b_i]$. The monopolist seeks to maximize his expected revenue $\int_D y(t) f(t)\, dt$ subject to incentive-compatibility constraints. Note that y is payment in this section.

This is similar to the problem addressed by Palfrey [19], except that Palfrey's monopolist had a fixed total number of units of each good to allocate. The present model can be interpreted as Palfrey's problem for the case of a single buyer, or equivalently there are several buyers and the monopolist can produce the goods at constant (zero) marginal cost without capacity constaints.

The discontinuities generated by the indivisibility of the goods mean that the optimal mechanism will be discontinuous. Hence the analysis of Section 2 does not help in solving this problem.

Let $Q(t)$ represent the optimized utility function, so that

$$\begin{aligned} Q(t) &= \max_s [t \cdot x(s) - y(s)] \\ &= t \cdot x(t) - y(t) \end{aligned} \tag{36}$$

and

$$Q'(t) = x(t) \tag{37}$$

whenever Q' is defined.

By Rochet [21], we have:

Lemma 1. *The mechanism $(x(t), y(t))$ is incentive compatible if and only if $t \cdot x(t) - y(t)$ is a convex function of t.*

It follows that the monopolist's revenue maximization problem is given by

$$\text{Max} \int_0^{b_n} \cdots \int_0^{b_1} [t \cdot Q'(t) - Q(t)]\, f(t)\, dt_1 \cdots dt_n \tag{38}$$

subject to

$$Q(t) \geqslant 0, \tag{39}$$

$$0 \leqslant \frac{\partial Q}{\partial t_i} \leqslant 1,^{10} \tag{40}$$

$$Q \text{ is convex.} \tag{41}$$

(In the objective function (38), use was made of (37). The constraint (39) is a free-exit or individual-rationality constraint; the constraint (40) recognizes that at most one unit of each good is to be sold to any buyer; and (41) follows from the lemma.)

Let $dt = dt_1 \cdots dt_n$, $dt_{-i} = dt_1 \cdots dt_{i-1}\, dt_{i+1} \cdots dt_n$, $t_{-i} = (t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)$, and $D_{-i} = \mathsf{X}_{j \neq i}[0, b_j]$. Then

$$\begin{aligned}
\int_D t_i \frac{\partial Q}{\partial t_i} f(t)\, dt &= \int_{D_{-i}} \int_0^{b_i} \frac{\partial Q}{\partial t_i} t_i f(t)\, dt_i\, dt_{-i} \\
&\qquad \text{(by Fubini's theorem)} \\
&= \int_{D_{-i}} \left\{ Q t_i f(t)|_0^{b_i} - \int_0^{b_i} Q \left[\frac{\partial}{\partial t_i} t_i f(t) \right] dt_i \right\} dt_{-i} \\
&\qquad \text{(by integration by parts)} \\
&= \int_{D_{-i}} Q(b_i, t_{-i})\, b_i f(b_i, t_{-i})\, dt_{-i} - \int_D Q \left(f(t) + t_i \frac{\partial f}{\partial t_i} \right) dt.
\end{aligned} \tag{42}$$

Thus the objective function (38) reduces to

$$\begin{aligned}
\text{Max} \sum_{i=1}^{n} \int_{D_{-i}} & b_i Q(b_i, t_{-i})\, f(b_i, t_{-i})\, dt_{-i} \\
& - \int_D [(n+1)\, f(t) + t \cdot f'(t)]\, Q(t)\, dt
\end{aligned} \tag{43}$$

subject to (39), (40), (41).

Examining (43), it is clear that if $(n+1)\, f(t) + t \cdot f'(t)$ is everywhere positive, which is a kind of hazard condition, it is in the seller's interest to minimize $Q(t)$, subject to boundary conditions. The next lemma shows how this is accomplished.

[10] $\partial Q/\partial t_i$ will exist a.e. Moreover, by convexity, left- and right-hand limits of $\partial Q/\partial t_i$ will exist in the interior of D and will satisfy (40).

LEMMA 2. *Consider* $Q: R^n \to R$ *satisfying* (39)–(41) *and*

$$Q(b_i, t_{-i}) = \beta_i(t_{-i}), \qquad i = 1, ..., n. \tag{44}$$

Then

$$\psi(t) = \max_{1 \leqslant i \leqslant n} \{0, t_i - b_i + \beta_i(t_{-i})\} \tag{45}$$

satisfies (39), (40), (41), *and* (44), *and* $\psi(t) \leqslant Q(t)$.

Proof. See the Appendix.

It can be seen from the modified objective function (43) that it is convenient to impose the regularity condition

$$(n+1) f(t) + t \cdot f'(t) \geqslant 0, \tag{46}$$

or equivalently

$$\frac{\partial}{\partial \lambda} \lambda^{n+1} f(\lambda t) \geqslant 0. \tag{47}$$

For any given set of $\beta_i: R^{n+1} \to R$, let

$$A_i = \{t \mid t_i - b_i + \beta_i(t_{-i}) \geqslant \max\{0, t_j - b_j + \beta_j(t_{-j})\}, \forall_j\}. \tag{48}$$

Then, using Lemma 2, the monopolist's problem reduces to

$$\max_{\beta_i} \sum_{i=1}^{n} \left\{ \int_{D_{-i}} b_i f(b_i, t_{-i})\, \beta_i(t_{-i})\, dt_{-i} - \int_{A_i} [(n+1) f(t) + t \cdot f'(t)](t_i - b_i + \beta_i(t_{-i}))\, dt \right\} \tag{49}$$

subject to

$$\beta_i(t_{-i}) \geqslant 0, \tag{50}$$

$$0 \leqslant \frac{\partial \beta_i}{\partial t_j} \leqslant 1, \tag{51}$$

and

$$\beta_i \text{ is convex.} \tag{52}$$

Let

$$P_i(\beta_i) = \int_{D_{-i}} \beta_i b_i f(b_i, t_{-i})\, dt_{-i} - \int_{A_i} [(n+1) f(t) + t \cdot f'(t)]\, \beta_i\, dt. \tag{53}$$

LEMMA 3. *If β_i^* maximizes* (49), *subject to* (50)–(52), *then* $P_i(\beta_i^*) \geqslant P_i(\beta_i)$ *for all* β_i *satisfying* (50), (51), (52).

Proof. See the Appendix.

Since

$$\frac{\partial \psi_j}{\partial t_i} \leqslant 1 = \frac{\partial \psi_i}{\partial t_i}, \tag{54}$$

there is a function $h_i(t_{-i})$ such that

$$A_i = \{t \mid t_i \geqslant h_i(t_{-i})\}. \tag{55}$$

Thus

$$P_i(\beta_i) = \int_{D_{-i}} \beta_i \left\{ b_i f(b_i, t_{-i}) - \int_{h_i(t_{-i})}^{b_i} [(n+1) f(t) + t \cdot f'(t)] \, dt_i \right\} dt_{-i}. \tag{56}$$

Although it appears that little can be said about the case of arbitrary numbers of goods, (56) provides a solution for the case $n = 2$, as we can see in the following lemma.

LEMMA 4. *Suppose β^* solves*

$$\max \int_0^b \beta(t) \, r(t) \, dt \tag{57}$$

subject to

$$\beta(0) \geqslant 0, \tag{58}$$

$$0 \leqslant \beta'(t) \leqslant 1, \tag{59}$$

and

$$\beta \text{ is convex.} \tag{60}$$

Let t_0 maximize $\int_{t_0}^b (t - t_0) \, r(t) \, dt$. Then

$$\beta^*(t) = \max\{0, t - t_0\}. \tag{61}$$

Proof. See the Appendix.

good case, given that (46) holds, it is never optimal for the seller to operate a stochastic policy: depending on his type, a buyer gets both goods, one, or none with certainty. (This generalizes the no-haggling result of Riley and Zeckhauser [20].) To find the optimal policy, we need only compare these four alternatives for any buyer type. Is it in the monopolist's interest to bundle, that is, to sell both goods as a package?

Since the optimal selling policy is nonstochastic, the monopolist can choose one of three pricing policies. First, he could simply price each commodity separately. Second, he could offer the goods for sale only as a bundle with a single bundle price. Third, he could offer to sell either separately or bundled, with a price for the bundle which is different from the sum of the separate prices. McAfee, McMillan, and Whinston [15] prove that the monopolist will never choose nonbundling; condition (46) guarantees that the third policy, mixed bundling, strictly dominates. Thus, when combined with the above result, for $n=2$, we have that the optimal selling mechanism requires setting three prices, for the goods individually and for the bundle.

5. Conclusion

Other problems with multidimensional asymmetries of information to which the techniques developed in this paper may be applicable include optimizing the structure of commodity taxation when taxpayers differ in several characteristics (examined by Mirrlees [17]); finding the welfare-maximizing regulatory rule when a regulatory agency knows neither the (constant) marginal cost nor the fixed cost of the regulated firm (posed in the concluding section of Baron and Myerson [2])[11]; and finding the welfare-maximizing pricing schedule for a multiproduct increasing-returns-to-scale firm when consumers' (multidimensional) types are unobservable (posed in the concluding section of Mirman and Sibley [16]).

Appendix

The following lemma was used in the proof of Theorem 1.

Lemma. *Suppose A and B are $n \times k$ matrices, with $n \leqslant k$ and rank $(B)=n$. Then $B^{\mathrm{T}}A$ is positive semidefinite if and only if $A=CB$ for some $n \times n$ positive semidefinite matrix C.*

[11] Lewis and Sappington [10] analyze optimal regulation of a monopolist when the regulator has imperfect information about the demand function as well as the firm's cost function.

Proof. (If): Suppose $A = CB$, with C positive semidefinite. Then $x^T B^T A x = (Bx)^T C(Bx) \geqslant 0$. Thus $B^T A$ is positive semidefinite.

(Only if): It is sufficient to show that the null space of B is contained in the null space of A [11]. Let $By = \mathbf{0}$ and $Bz \neq \mathbf{0}$. Let $x = y + \delta z$.

$$\begin{aligned} h(\delta) &= x^T B^T A x \\ &= (Bx)^T A x \\ &= (\delta Bz)^T A(y + \delta z) \\ &= \delta (Bz)^T A y + \delta^2 (Bz)^T A z. \end{aligned}$$

Now $h(\delta) \geqslant 0$ and $h(0) = 0$. Thus

$$0 = h'(0) = (Bz)^T A y.$$

Since B is of rank n, the set of vectors in the form Bz is R^n. Thus $Ay = 0$ for all y satisfying $By = \mathbf{0}$. That is, the null space of B is contained in the null space of A, and there is a C such that $A = CB$. C is positive semidefinite since $0 \leqslant x^T B^T A x = (Bx)^T CBx$ and $\{Bx\}$ spans R^n. Q.E.D.

We prove now the lemmas used in Section 4.

Proof of Lemma 2. Let

$$\psi_i(t) = \max\{0, t_i - b_i + \beta_i(t_{-i})\}. \tag{A1}$$

Since Q satisfies (37), (38), (39), β_i does as well, and hence ψ_i does. Thus $\psi(t) = \max \psi_i(t)$ satisfies (39)–(41) (recall that the maximum of convex functions is convex). Furthermore,

$$\begin{aligned} Q(t) &\geqslant Q(b_i, t_{-i}) + Q'(t) \cdot (t - (b_i, t_{-i})) \\ &= \beta_i(t_{-i}) + \frac{\partial Q(t)}{\partial t_i}(t_i - b_i) \\ &\geqslant \beta_i(t_{-i}) + (t_i - b_i) \\ &= \psi_i(t). \end{aligned} \tag{A2}$$

Thus $Q(t) \geqslant \max_i \psi_i(t) = \psi(t)$. Q.E.D.

Proof of Lemma 3. The set of functions satisfying (48), (49), and (50) is a convex set of functions, so if β_i and β_i^* are in the set, then $\lambda\beta_i + (1 - \lambda)\beta_i^*$

$$\rho(\lambda)=\sum_{i=1}^{n}\left\{\int_{D_{-i}} b_i f(b_i, t_{-i})(\lambda\beta_i+(1-\lambda)\beta_i^*)\,dt_{-i}\right.$$

$$\left.-\int_{A_i}[(n+1)\,f(t)+t\cdot f'(t)](t_i-b_i+\lambda\beta_i+(1-\lambda)\beta_i^*)\,dt\right\}. \qquad \text{(A3)}$$

Then, if β_i^* is optimal, we have $\rho'(0)\leqslant 0$, or

$$\begin{aligned} 0 &\geqslant \int_{D_{-i}} (\beta_i-\beta_i^*)\,b_i f(b_i, t_{-i})\,dt_{-i} \\ &\quad -\int_{A_i}[(n+1)\,f(t)+t\cdot f'(t)](\beta_i-\beta_i^*)\,dt \\ &\quad -\sum_{j=1}^{n}\int_{\partial A_j}[(n+1)\,f(t)+t\cdot f'(t)]\,\psi_j(t)\,dt \\ &= \int_{D_{-i}} (\beta_i-\beta_i^*)\,b_i f(b_i, t_{-i})\,dt_{-i} \\ &\quad -\int_{A_i}[(n+1)\,f(t)+t\cdot f'(t)](\beta_i-\beta_i^*)\,dt, \end{aligned} \qquad \text{(A4)}$$

where ∂A_j indicates the oriented surface integral on the surface of A_j (Edwards [5]), and A_j is evaluated at β_i^*, since $\lambda=0$. The terms ∂A_j drop out because, at the borders of A_j, either $\psi_i=\psi_j$ or $\psi_i=0$. Thus, by Stokes's theorem,

$$\begin{aligned} &\int_{\partial A_i}[(n+1)f(t)+t\cdot f'(t)]\,\psi_i(t)\,dt \\ &\quad =\sum_{j\neq i}\int_{\partial A_j}[(n+1)\,f(t)+t\cdot f'(t)]\,\psi_j(t)\,dt. \end{aligned} \qquad \text{(A5)}$$

Let

$$P_i(\beta_i)=\int_{D_{-i}}\beta_i b_i f(b_i, t_{-i})\,dt_{-i}-\int_{A_i}[(n+1)\,f(t)+t\cdot f'(t)]\,\beta_i\,dt, \qquad \text{(A6)}$$

where A_i is evaluated for β_i^*. Then a necessary condition for β_i^* to be optimal is, for all β_i satisfying (48), (49), (50), that $P_i(\beta_i)\leqslant P_i(\beta_i^*)$, from (A4).
Q.E.D.

Proof of Lemma 4. If $\int_0^b r(t)\,dt>0$, then there is clearly no solution. Thus $\int_0^b r(t)\,dt\leqslant 0$, and we may presume $\beta^*(0)=0$. Define $R(t)=\int_t^b r(s)\,ds$.

Integrate by parts:

$$\int_0^b \beta(t)\, r(t)\, dt = \int_0^b \beta'(t)\, R(t)\, dt. \tag{A7}$$

Note that, by integrating by parts,

$$\int_{t_0}^b R(t)\, dt = \int_{t_0}^b (t - t_0)\, r(t)\, dt. \tag{A8}$$

Thus t_0 maximizes $\int_{t_0}^b R(t)\, dt$, by (A8) and the definition of t_0. Thus, if β satisfies (57), (58), (59),

$$\int_0^b \beta'(t)\, R(t)\, dt = \int_0^{t_0} \beta'(t)\, R(t)\, dt + \int_{t_0}^b \beta'(t)\, R(t)\, dt$$

$$= \alpha_0 \int_0^{t_0} R(t)\, dt + \alpha_1 \int_{t_0}^b R(t)\, dt$$

(using the Intermediate Value Theorem)

$$\leqslant 0 \int_0^{t_0} R(t)\, dt + 1 \int_{t_0}^b R(t)\, dt$$

$$= \int_0^b \beta^*(t)\, r(t)\, dt. \tag{A9}$$

(The inequality follows because $\int_0^b R(t)\, dt \leqslant \int_{t_0}^b R(t)\, dt$, $\alpha_0, \alpha_1 \in [0, 1]$, and $0 = \int_b^b R(t)\, dt \leqslant \int_{t_0}^b R(t)\, dt$, since t_0 maximizes $\int_t^b R(s)\, ds$.) Q.E.D.

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