Before going to Europe on business, a man drove his Rolls-Royce to a downtown NY City bank and went in to ask for an immediate loan of $5,000. The loan officer, taken aback, requested collateral. "Well, then, here are the keys to my Rolls-Royce", the man said. The loan officer promptly had the car driven into the bank's underground parking for safe keeping, and gave him $5,000.

Two weeks later, the man walked through the bank's doors, and asked to settle up his loan and get his car back. "That will be $5,000 in principal, and $15.40 in interest", the loan officer said. The man wrote out a check and started to walk away.

"Wait sir", the loan officer said, "while you were gone, I found out you are a millionaire. Why in the world would you need to borrow $5,000?"

The man smiled. "Where else could I park my Rolls-Royce in Manhattan for two weeks and pay only $15.40?"

---

**Investment**

**Net Present Value**

- Stream of payments \( A_0, A_1, \ldots \)

\[
NPV = A_0 + \frac{A_1}{1+r} + \frac{A_2}{(1+r)(1+r_2)} + \frac{A_3}{(1+r)(1+r_2)(1+r_3)} + \ldots
\]

- Consol: same payment forever

\[
v = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots = \frac{1}{1-r}
\]

- Common interest rate \( r \)
Mortgage

- Pay $1 per period, $n$ periods

\[
NPV = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots + \frac{1}{(1+r)^n}
\]

\[
= \frac{1}{r} \left( \frac{1}{1-(1+r)^{-n}} \right)
\]

Car Loan

- \( r = \frac{1}{2}\% \) per month
- $1/ month for 60 months =

\[
NPV = \frac{1}{.005} \left( 1 - \frac{1}{(1.005)^{60}} \right) = 51.73
\]

- To borrow $20,000 requires

\[
\frac{20,000}{51.73} = $386.66 / \text{mo.}
\]

Value of the Lottery

- $1M/yr, 20 years

\[
PV = 1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^{19}} = 1 + \frac{1}{r} \left( 1 - \frac{1}{(1+r)^{19}} \right)
\]

<table>
<thead>
<tr>
<th>$r$</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV (000s)</td>
<td>$15,324</td>
<td>$14,134</td>
<td>$13,085</td>
<td>$12,158</td>
<td>$11,336</td>
<td>$9,365</td>
</tr>
</tbody>
</table>


Bond Price

- Bond pays $1000 in 10 years

\[
\text{Price} = \text{NPV} = \frac{1000}{(1 + r)^{10}}
\]

- Interest rate increase causes bond prices to fall

MBA Investment Strategy

- Compute NPV
- Undertake project if NPV > 0
- Preferable to calculating internal rates of return (solve equation NPV=0 for \( r \)) because IRR not well defined

Investment Under Uncertainty

MBA Strategy:
- Use risk-adjusted interest rate
- Risk adjustment for project, not parent!
  - Ibbotson Cost of Capital Yearbook
- Compute expected NPV
- Undertake if E NPV > 0
Rates of Return

If you had invested $1 in the following from end of 1925 to end of 1999 it would have increased to

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Annual Return</th>
<th>Ending Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>11.3%</td>
<td>$2,845.63</td>
</tr>
<tr>
<td>Small company stock index</td>
<td>12.6%</td>
<td>$6,640.79</td>
</tr>
<tr>
<td>Long-term corporate bond index</td>
<td>5.6%</td>
<td>$56.38</td>
</tr>
<tr>
<td>Long-term government bond index</td>
<td>5.1%</td>
<td>$40.22</td>
</tr>
<tr>
<td>Intermediate-term government bond index</td>
<td>5.2%</td>
<td>$43.93</td>
</tr>
<tr>
<td>U.S. Treasury Bills</td>
<td>3.8%</td>
<td>$15.64</td>
</tr>
<tr>
<td>Inflation</td>
<td>3.1%</td>
<td>$9.39</td>
</tr>
</tbody>
</table>

Risk and Return

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Geometric Mean</th>
<th>Standard Deviation</th>
<th>Arithmetic Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small company stocks</td>
<td>12.6%</td>
<td>33.6%</td>
<td>17.6%</td>
</tr>
<tr>
<td>Large company stocks</td>
<td>11.3%</td>
<td>20.1%</td>
<td>13.3%</td>
</tr>
<tr>
<td>Long-term corporate bonds</td>
<td>5.6%</td>
<td>8.7%</td>
<td>5.9%</td>
</tr>
<tr>
<td>Long-term government bonds</td>
<td>5.1%</td>
<td>9.3%</td>
<td>5.5%</td>
</tr>
<tr>
<td>Intermediate-term gov't bonds</td>
<td>5.2%</td>
<td>5.6%</td>
<td>5.4%</td>
</tr>
<tr>
<td>U.S. Treasury Bills</td>
<td>3.8%</td>
<td>3.2%</td>
<td>3.8%</td>
</tr>
<tr>
<td>Inflation</td>
<td>3.1%</td>
<td>4.5%</td>
<td>3.2%</td>
</tr>
</tbody>
</table>

Mean Returns

- Geometric mean: \( \left( \prod_{i=1}^{n} a_i \right)^{1/n} \)
- Arithmetic mean: \( \frac{1}{n} \sum_{i=1}^{n} a_i \)

\[
\log \left( \prod_{i=1}^{n} a_i \right)^{1/n} = \frac{1}{n} \sum_{i=1}^{n} \log a_i = \log \left( \frac{1}{n} \sum_{i=1}^{n} a_i \right)
\]
Mean Returns, Continued

- Geometric makes sense when using rates of return over several years
- Arithmetic would be used for expected returns in a given year

Option Value

- NPV is appropriate for “now or never” decisions
- Now or later requires an additional consideration
  - Sell a painting
  - Drill an oil well
  - Build a factory
- Investment destroys option to invest

Example

- Spend $C < 1$ to produce a value $V$
- $V \sim U[0,1]$, interest rate $r$
- Use cutoff value $V_0$: invest if $V \geq V_0$
- Produces $\text{NPV} = J(V_0)$

\[
J(V_0) = (1 - V_0) \left( \frac{1 + V_0}{2} - C \right) + V_0 \left( \frac{1}{1 + r} J(V_0) \right).
\]
Investment Value

\[ J(V_0) = \frac{(1-V_0)\left(\frac{1+V_0}{2} - C\right)}{1-\frac{V_0}{1+r}}. \]

- Maximized when

\[ V_0 = (1 + r) - \sqrt{r^2 + 2r(1 - C)}. \]

---

Resource Extraction

- Fixed supply of a resource \( R \)
- Constant demand elasticity \( \varepsilon \)
- Let \( Q_t \) represent the quantity consumed at time \( t \).
- Arbitrage: price rises at interest rate

\[ aQ_0^{-\varepsilon}(1+r)^t = p(Q_0)(1+r)^t = p(Q_t) = aQ_t^{-\varepsilon} \]
Resource Use

\[ R = (Q_0 + Q_1 + Q_2 + \ldots) \]
\[ = Q_0 \left[ 1 + (1 + r)^{-\varepsilon} + (1 + r)^{-2\varepsilon} + \ldots \right] = \frac{Q_0}{1 - (1 + r)^{-\varepsilon}} \]

- Arbitrage spreads use out
  - Never run out
- Prices rise at interest rate
- Markets don’t view natural resources this way
  - reflecting alternatives, technological change
- \( r = 0.05, \varepsilon = 2, \quad 1 - (1 + r)^{-\varepsilon} = 9.3\% \) annual
  - Half life 7 years

Tree-Cutting

- Time to harvest then replant
  - Trees, lobsters, fish, cows
- Value after growth \( t \) of \( b(t) \)
- Continuous time interest rate
  \[ e^{-\delta t} = \left( \frac{1}{1 + r} \right)^t \]

NPV

\[ e^{-\delta T} pb(T) + e^{-2\delta T} pb(T) + e^{-3\delta T} pb(T) + \ldots \]
\[ = \frac{e^{-\delta T} pb(T)}{1 - e^{-\delta T}} = \frac{pb(T)}{e^{\delta T} - 1} \]

- FOC \[ \frac{b'(T)}{b(T)} = \frac{\delta}{1 - e^{-\delta T}} \]
Tree-Cutting

- Ramsey Rule: cut down the trees when they are growing at the interest rate
  \[
  \frac{b'(T)}{b(T)} = \frac{\delta}{1 - e^{-\delta T}}
  \]
- Approximately correct
- US policy of maximum sustainable yield sends \( \delta \to 0 \), yields \( b'(T) \to \frac{1}{T} \)

Collectibles

- time t starts 0 to \( \infty \)
- Quantity supplied \( q(t) = q_0 e^{-\delta t} \)
- \( \epsilon \) = elasticity of demand
- \( g \) = growth rate of population
- \( r \) = discount rate (e^{-rt})
Demand and Supply

• Demand \( x_d(p,t) = ae^{gt} p^{-\epsilon} \)
• Demand and supply equate to give the marginal use value of an owner at time \( t \)
  \[ q_0 e^{-St} = q(t) = x_d(v,t) = ae^{gt} v^{-\epsilon} \]
  or
  \[ v = \left( \frac{a}{q_0} \right)^{\frac{1}{\epsilon}} e^{\delta g t} \]

Determination of Price

• Marginal holder must be just indifferent to holding
• Marginal holder who buys at \( t \) and sells at \( t+\Delta \) gets
  \[ \Delta \int_0^\Lambda e^{-\tau s}(v-s)du - p(t) + e^{-\tau \Lambda} e^{-\delta \Lambda} p(t+\Lambda) \]
Per Period Value of Holding

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \int_0^\Delta e^{-r\Delta}(\nu - s)du - \frac{p(t)}{\Delta} e^{-(r+\delta)\Delta} p(t + \Delta)
\]

\[
= \lim_{\Delta \to 0} \nu - s + \frac{p(t + \Delta) - p(t)}{\Delta} - \frac{1 - e^{-(r+\delta)\Delta}}{\Delta} p(t + \Delta)
\]

\[
= \nu - s + p'(t) - (r + \delta)p(t)
\]

Marginal Owner is Indifferent

- Marginal owner
  \[
  v = \left( \frac{a}{q_0} \right)^{\frac{1}{\zeta}} \frac{\delta + g}{e^{\zeta}}
  \]

- Yields
  \[
  p'(t) = (r + \delta)p(t) + s - (r + \delta)p(t) + s - \left( \frac{a}{q_0} \right)^{\frac{1}{\zeta}} \frac{\delta + g}{e^{\zeta}}
  \]

Solution

\[
p(t) = e^{(r+\delta)t} \left( p(0) + \frac{1 - e^{-(r+\delta)t}}{(r+\delta)} - \left( \frac{a}{q_0} \right)^{\frac{1}{\zeta}} \frac{1 - e^{-\left( r + \delta - \frac{\delta + g}{e^{\zeta}} \right)}}{r + \delta - \frac{\delta + g}{e^{\zeta}}} \right)
\]
Necessary Conditions

• Present value of marginal use value is finite:
  \[ r + \delta - \frac{\delta + g}{\epsilon} > 0 \]

• Not everyone wants to own the good:
  \[ \lim_{t \to \infty} e^{-rt} p(t) < \infty, \]

Starting Price

• Either
  \[ p(0) = \left( \frac{a}{q_0} \right)^{\frac{1}{\epsilon}} \frac{1}{r + \delta - \frac{\delta + g}{\epsilon}} \frac{1}{(r + \delta)^s} \geq 0 \]
  or
  \[ \alpha = p(0) = \left( \frac{a}{q(0)} \right)^{\frac{1}{\epsilon}} \frac{1}{r + \delta - \frac{\delta + g}{\epsilon}} \frac{1}{(r + \delta)^s} \]

and \( q(0) \leq q_0 \)

Implications

• May destroy some quantity initially
• Price rises exponentially
• Storage costs enter linearly

\[ p(t) = \left( \frac{a}{q(t)} \right)^{\frac{1}{\epsilon}} \frac{e^{\frac{\delta + g}{\epsilon} \frac{t}{\epsilon}}}{r + \delta - \frac{\delta + g}{\epsilon}} = \frac{s}{r + \delta} \]